Partial inconsistency and bitopology

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Electronic coordinates

These slides are linked from my page on partial inconsistency and vector semantics of programming languages:

http://www.cs.brandeis.edu/∼bukatin/partial_inconsistency.html

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Abstract

Bitopology occurs in at least three different ways in the context of partial inconsistency: via bitopological Stone duality and d-frames, via group dual topologies and pairwise continuity, and via Rodabaugh representation of L-fuzzy bitopologies as fuzzy topologies valued in L^2 bilattices.

Partially inconsistent interval numbers, $[R]$, are obtained by adjoining formally inconsistent elements, namely formal pseudosegments $[a, b]$ with the contradictory property that $b < a$, to the ordinary interval numbers. Unlike ordinary interval numbers, $[R]$ is a group and a vector space. The first mention known to us is by Warmus in 1956, and there were multiple rediscoveries.

We show that $[R]$ with the added "infinity crust" is isomorphic to the d-frame of the (lower, upper) bitopology on reals, discuss the roles $[R]$ is playing in the context of group dual topologies and pairwise continuity of the anti-monotonic group negation, introduce a natural $[R]$-valued partial metric on $[R]$ itself, discuss the properties of $[R]$ as a bilattice, and note that real-valued fuzzy bitopologies are fuzzy topologies valued in $[R]$.

Partial inconsistency landscape

A variety of different studies seem to fall together like pieces of a single puzzle. Here are some of the puzzle connectors:

- Bilattices
- Hahn-Jordan decomposition
- Partial inconsistency
- Non-monotonic inference
- Bitopology
- Negative distance, negative degree of set membership, negative probability
- Group and vector space semantics of programming languages compatible with Scott domain semantics

Outline

Bicontinuous domains

Klaus Keimel, Bicontinuous Domains and Some Old Problems in Domain Theory, Electronic Notes in Theoretical Computer Science, **257**, 3 December 2009, p. 35–54.

Bicontinuous domains – continuous both with respect to $\sqsubseteq$ and $\sqsupseteq$.

Two topologies: Scott topology and dual Scott topology (Scott topology induced by $\sqsupseteq$). Their join is called "bi-Scott".

Duality and the domain of arrows

Bicontinuity allows for a very convenient notion of dual domain, D^{Op}: just flip the order.

Then given a domain D, one can define its domain of arrows: $D^{Op} \times D$.

Intuition: an arrow is larger if its initial end is smaller ("contravariance with respect to the initial end").

Quasi-metrics on D are monotonic (and typically Scott continuous) over $D^{Op} \times D$.

The domain of arrows as a bilattice

$D^{Op} \times D$ produces the informational partial order, $\sqsubseteq$.

$D \times D$ produces the material partial order, $\leq$.

The domain of arrows and profunctors (distributors)

More generally, one can consider arrows from domain C to D.

The domain of such arrows is $C^{Op} \times D$.

From here it is a straight road to profunctors (also known as distributors or modules).

Interval numbers

Segments $[a, b]$ on real line, $a \leq b$.

What $[a, b]$ means: $[a, b]$ stands for a partially defined number x, what is known about x is the constraint $a \leq x \leq b$.

Partial order on the interval numbers:
$[a, d] \sqsubseteq [b, c]$ iff $a \leq b$ $(\leq)$ $c \leq d$.

Here $[b, c]$ is better (more precisely) defined than $[a, d]$.

Addition and weak minus

Addition: $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$.

Weak minus: $-[a, b] = [-b, -a]$.

These are monotonic operations:
$x \sqsubseteq y \Rightarrow x + z \sqsubseteq y + z$ and $-x \sqsubseteq -y$.

However, the minus is weak, e.g. $-[2, 3] = [-3, -2]$, so
$-[2, 3] + [2, 3] = [-1, 1] \sqsubset [0, 0]$.

So one does not get a group here.
And it would be nice to have a group.

Partially inconsistent interval numbers

Add pseudosegments $[a, b]$, such that $b < a$.

This corresponds to contradictory constraints, $x \leq b \& a \leq x$.

The new set consists of segments and pseudosegments.

Addition: $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$.

True minus: $-[a, b] = [-a, -b]$.

$-[a, b] + [a, b] = [0, 0]$.

This gets us a group.

True minus is antimonotonic

$x \sqsubseteq y \Rightarrow -y \sqsubseteq -x.$

True minus maps precisely defined numbers, $[a, a]$, to precisely defined numbers, $[-a, -a]$.

Other than that, true minus maps segments to pseudosegments and maps pseudosegments to segments.

In the bicontinuous setup, true minus is a bicontinuous functon from $[R]$ to $[R]^{Op}$ (or from $[R]^{Op}$ to $[R]$).

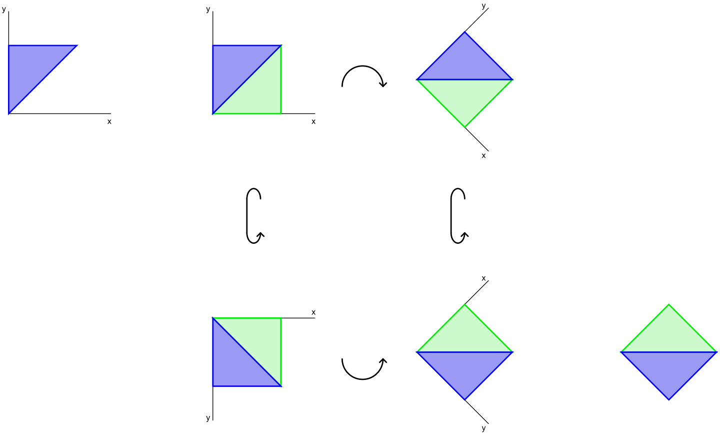
Multiple rediscoveries

Known under various names: Kaucher interval arithmetic, directed interval arithmetic, generalized interval arithmetic, modal interval arithmetic, interval algebraic extensions, etc.

First mention we know: M. Warmus, Calculus of Approximations. Bull. Acad. Pol. Sci., Cl. III, 4(5): 253-259, 1956, http://www.cs.utep.edu/interval-comp/warmus.pdf

A comprehensive repository of literature on the subject is maintained by Evgenija Popova: The Arithmetic on Proper & Improper Intervals (a Repository of Literature on Interval Algebraic Extensions), http://www.math.bas.bg/∼epopova/directed.html

From Cartesian to Hasse representation

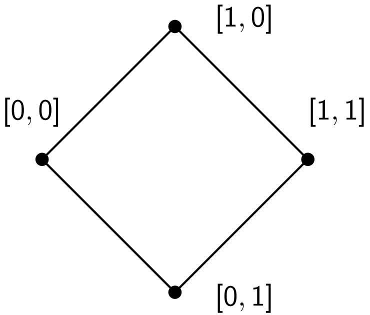


Partially inconsistent interval numbers as a domain of arrows

$$[R] = \mathbb{R} \times \mathbb{R}^{Op}$$

(There is a tension between the group structure on $\mathbb{R}$ and $[R]$ and the axioms of domains requiring $\bot$ and $\top$ elements which can be satisfied by restricting to a segment of reals, or by adding $-\infty$ and $+\infty$. I am mostly being ambiguous about this in this slide deck, but this is something to keep in mind.)

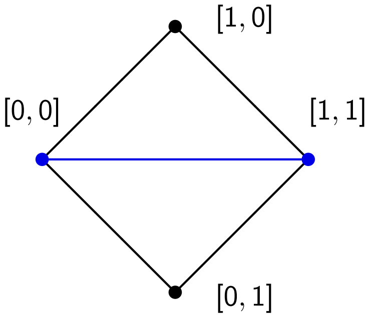
Partially inconsistent interval numbers within a segment



$[a, b] \leq [c, d]$ iff $a \leq c, b \leq d$

$[a, d] \sqsubseteq [b, c]$ iff $a \leq b, c \leq d$

Partially inconsistent interval numbers within a segment



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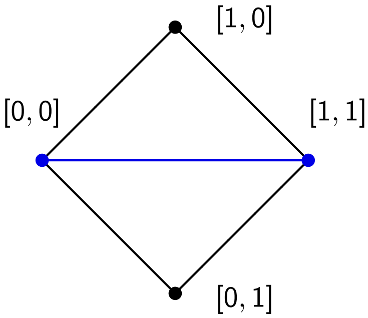
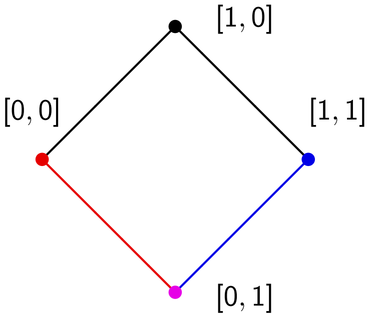
Partially inconsistent interval numbers within a segment



blue – precisely defined numbers

pseudosegments are above the blue

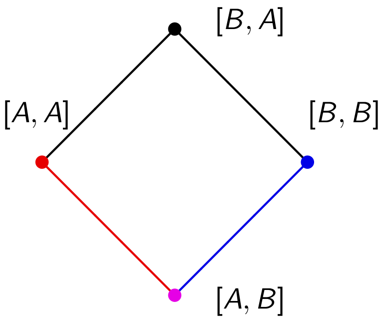
Negative and positive subspaces



Negative – space of upper bounds $[0, x]$

Positive – space of lower bounds $[x, 1]$

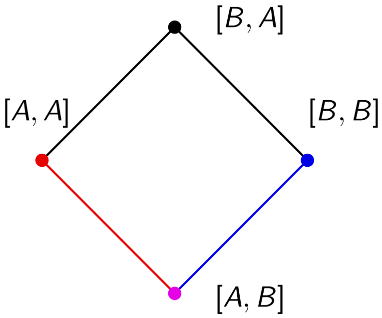
Negative and positive subspaces



Negative – space of upper bounds $[A, x]$

Positive – space of lower bounds $[x, B]$

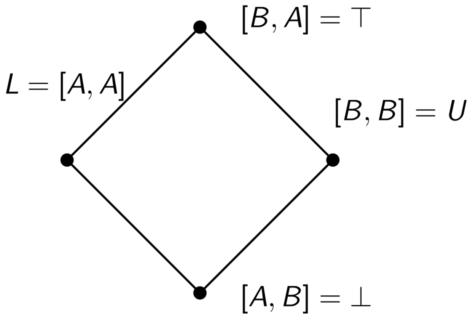
Negative and positive subspaces



We require $A < B$.

We can even allow $A = -\infty, B = \infty$.

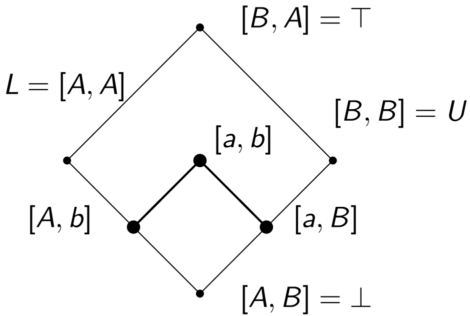
Negative and positive subspaces



We require $A < B$.

We can even allow $A = -\infty, B = \infty$.

Decomposition into negative and positive subspaces



$$[A, b] = [a, b] \wedge \bot = [a, b] \sqcap L$$
$$[a, B] = [a, b] \vee \bot = [a, b] \sqcap U$$
$$[a, b] = [A, b] \sqcup [a, B]$$

Bitopology and d-frames

Achim Jung, M. Andrew Moshier. On the bitopological nature of Stone duality. Technical Report CSR-06-13. School of Computer Science, University of Birmingham, December 2006, 110 pages.

This text has a lot of very interesting material. I am only touching a bit of it here.

d-frames

Take two frames L_+ and L_- (the informal intent is for their elements correspond to open sets where the predicates are true and where they are false).

$L = L_+ \times L_-$ is a bilattice.

Introduce $Con, Tot \subseteq L$ with the informal intent that for pairs of open sets $U = \langle U_+, U_- \rangle$, $U \in Con$ when $U_+ \cap U_- = \emptyset$, and $U \in Tot$ when $U_+ \cup U_-$ covers the whole space.

This allows to handle partial inconsistency and the bilattice pattern does appear. (L_+, L_-, Con, Tot) is called a d-frame.

Bitopological Stone duality

This paper studies Stone duality modified to apply to bitopological spaces and d-frames.

It also demonstrates that a number of classical dualities, namely dualities of Stone, Ehresmann-Bénabou, and Jung-Sünderhauf, actually have bitopological nature, namely they are special cases of the Stone duality between bitopological spaces and d-frames.

d-frame for the (lower, upper) bitopology on $\mathbb{R}$

d-frame elements are pairs $\langle L, U\rangle$ of open rays, $\langle(-\infty, a), (b, +\infty)\rangle$ (a and b are allowed to take $-\infty$ and $+\infty$ as values).

Non-overlapping pairs of open rays are consistent ($a \leq b$), overlapping pairs of open rays ($b < a$) are total.

Correspondence with partially inconsistent interval numbers

The bilattice isomorphism between d-frame elements and partially inconsistent interval numbers with "infitinity crust": $\langle(-\infty, a), (b, +\infty)\rangle$ corresponds to a partially inconsistent interval number $[a, b]$.

Consistent, i.e. non-overlapping, pairs of open rays ($a \leq b$) correspond to segments. Total, i.e. covering the whole space, pairs of open rays ($b < a$) correspond to pseudosegments.

Pseudosegments and negative set membership

Consider $\mathbb{R}$ as a characteristic function, and subtract from it characteristic functions of $(-\infty, a)$ and $(b, +\infty)$.

If $a \leq b$, we get the usual charactertic function for $[a, b]$.

However if $b < a$, we get the generalized characteristic function which takes value -1 on (a, b) and 0 elsewhere.

Topological asymmetry

Algebraically we can say that totally defined numbers $[a, a]$ belong to both segments and pseudosegments, or to neither.

But topologically (and via characteristic functions), this symmetry must be broken.

We brake it in favor of the "natural" viewpoint: totally defined numbers are segments, and not pseudosegments.

But one can brake it in favor of the dual viewpoint, by considering dual d-frames of closed sets (and stipulating that characteristic functions of segments take value 1 only on their interiors).

Another route to bitopology

A bitopology with two specialization orders pointing in the opposite directions is what seems to be required to handle antimonotonic functions well.

The true negation would typically be a pairwise continuous function from (X, T, T^{-1}) to (X, T^{-1}, T),
where T^{-1} is a group dual topology of T.

S. Andima, R. Kopperman, P. Nickolas, An Asymmetric Ellis Theorem, Topology and Its Applications **155**, 146–160 (2007)

Antimonotonic bicontinuous group inverse

$- : \mathbb{R} \to \mathbb{R}$.

The minus is bicontinuous from the (lower, upper) bitoplogy to the (upper, lower) bitopology and vice versa.

The corresponding inverse image map between the d-frames is very similar to the weak minus on $[R]$ (Ginsberg involution), except that the order of bitopological components also needs to be swapped to respect bitopological duality ($\mathbb{R} \times \mathbb{R}^{Op} \leftrightarrow \mathbb{R}^{Op} \times \mathbb{R}$).

Antimonotonic bicontinuous group inverse

In a simular fashion, the true minus operation on $[R]$ is bicontinuous between a (T, T^{-1}) bitopology on $[R]$ and its dual (T^{-1}, T) bitopology, and vice versa.

Here T and T^{-1} must be group dual topologies of each other.

The main case: T is the Scott topology corresponding to $\sqsubseteq$, and T^{-1} is the Scott topology corresponding to $\sqsupseteq$.

Rodabaugh representation

S. Rodabaugh, Functorial Comparisons of Bitopology with Topology and the Case for Redundancy of Bitopology in Lattice-valued Mathematics, Applied General Topology **9**(1), 77–108 (2008)

L-valued bitopology can be understood as L^2-valued topology, and, in particular, that ordinary bitopology can be understood as 4-valued topology. The 4-valued set here is the standard bilattice of 4 elements playing the same role in bitopology as the Sierpinski space plays in topology.

The L^2 in general is also a bilattice, with $\sqsubseteq$ being obtained from the product $(L, \sqsubseteq) \times (L, \sqsubseteq)$ and the material order, $\leq$, being obtained from the product of the dual lattice by the original one, $(L, \sqsupseteq) \times (L, \sqsubseteq)$.

Real-valued bitopology

Hence fuzzy bitopology valued in $\mathbb{R}$ can be represented as a fuzzy topology valued in $[R]$.

Lawvere duality between distances and relations

Order-reversing isomorphisms.

E.g. 0 is the smaller element for distances vs. 0 is the maximal degree of equality for relations.

Or fuzzy equalities into (Heyting) algebra of open sets for Ω-sets vs. partial ultrametrics into (Brouwerian) algebra of closed sets.

E.g. Bukatin, Kopperman, Matthews, Some Corollaries of the Correspondence between Partial Metrics and Multivalued Equalities, In press, Fuzzy Sets and Systems (2013), online publication: http://dx.doi.org/10.1016/j.fss.2013.08.016

Lawvere duality for the domain of arrows

$D^{Op} \times D \leftrightarrow D \times D^{Op}$

We have already seen this in the context of (antimonotonic bicontinuous) group inverse operations.

There are other contexts of order-reversing isomorphisms, where we see this.

Negative self-distance

The standard partial metric on the interval numbers is $p([a_1, b_1], [a_2, b_2]) = \max(b_1, b_2) - \min(a_1, a_2)$.

Hence the self-distance for $[a, b]$ is $b - a$.

If we extend this formula to pseudosegments, the self-distance of pseudosegments turns out to be negative.

Weak vs strong axioms

Partial metrics can be understood as upper bounds for “ideal distances”.

One often has to trade the tightness of those bounds for nicer sets of axioms.

E.g. the natural upper bound for the distance between $[0, 2]$ and $[1, 1]$ is 1, and there is a weak partial metric which yields that.

However, if one wants to enjoy the axiom of small self-distances, $p(x, x) \leq p(x, y)$, one has to accept $p([0, 2], [1, 1]) = 2$, since $p([0, 2], [0, 2]) = 2$.

Lower bounds

A similar trade can be made for lower bounds. The standard interval-valued relaxed metric produces the gap between non-overlapping segments as their lower bound, but takes 0 as the lower bound for the distance between overlapping segments (hence 0 is also the lower bound for self-distance).

If one settles for a less tight lower bound and allows the lower bound to be negative in those cases, one can obtain a distance with much nicer properties:

$l([a_1, b_1], [a_2, b_2]) = \max(a_1, a_2) - \min(b_1, b_2)$.

An $[R]$-valued distance on $[R]$

We think about the pair $\langle l, p \rangle$ as a relaxed metric valued in $[R]$.

The self-distance of $[a, b]$ is $[a - b, b - a]$ and the self-distance of a pseudosegment is a pseudosegment.

The map $[a, b] \mapsto [b, a]$ expressing the symmetry between segments and pseudosegments also transforms $\langle l, p \rangle$ into $\langle p, l \rangle$.

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