Remarks on Partially Inconsistent Interval Numbers

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1 Introduction

Interval numbers are segments $[a,b]$ on the real line where $a \leq b$. *Partially inconsistent interval numbers* are obtained by adding *pseudosegments* $[a,b]$ with the contradictory property that $b < a$.

This structure was independently discovered many times and is known under various names including Kaucher interval arithmetic, directed interval arithmetic, generalized interval arithmetic, and modal interval arithmetic (a comprehensive repository of literature on the subject is maintained by Evgenija Popova [5]). The first mention known to us is by Warmus in 1956 [7].

While interval numbers don't form a group with respect to addition, partially inconsistent interval numbers do form a group. The negation operation in that group is antimonotonic with respect to the partial order $\sqsubseteq$ ($[a,d] \sqsubseteq [b,c]$ iff $a \leq b$ and $c \leq d$).

When one allows a and b to also take values of $-\infty$ and $+\infty$, partially inconsistent interval numbers form a bilattice and are isomorphic to the d-frame of the (lower, upper) bitopology on reals.

Antimonotonic negation is related to the notion of bitopological group.

Signed measures (a measure of a set is allowed to be negative) and signed multisets (an element is allowed to have a negative degree of membership) appear naturally in this context. Partial metrics are allowed to take negative values in this context and relaxed metrics are themselves valued in partially inconsistent interval numbers.

Partially inconsistent interval numbers have interesting symmetries: between segments and pseudosegments, between upper and lower bounds (Ginsberg involution), and between $\sqsubseteq$ and $\leq$.

The bilattice of partially inconsistent interval numbers seems to play a fundamental role in mathematics of partial inconsistency.

2 Addition and Weak Minus

Addition on interval numbers (and partially inconsistent interval numbers) is defined component-wise: $[a_1,b_1]+[a_2,b_2]=[a_1+a_2,b_1+b_2]$. The operation of weak minus is defined as $-[a,b]=[-b,-a]$. These operatons are monotonic with respect to $\sqsubseteq$.

Consider $-[a,b]+[a,b]=[-b,-a]+[a,b]=[a-b,b-a]$. If $a<b$, then the strict inequality, $[a-b,b-a]\sqsubset[0,0]$, holds. So if $a<b$, $-[a,b]+[a,b]$ approximates $[0,0]$, but is not equal to it, hence interval numbers with weak minus don't form a group.

3 Pseudosegments and True Minus

If one allows pseudosegments, one can define the component-wise "true minus": $-[a,b]=[-a,-b]$. Partially inconsistent interval numbers with the component-wise addition and the true minus form a group.

The true minus maps precisely defined numbers, $[a,a]$, to precisely defined numbers, $[-a,-a]$. Other than that, the true minus maps segments to pseudosegments and maps pseudosegments to segments. The true minus is antimonotonic with respect to $\sqsubseteq$.

4 Bilattice Properties

There are two partial orders on partially inconsistent interval numbers. The "informational order", $\sqsubseteq$, is defined above. The "material order" is component-wise: $[a,b]\leq[c,d]$ iff $a\leq c$ and $b\leq d$.

The original definition of bilattice given by Ginsberg [3] also calls for each of those orders to form a complete lattice. If we want to satisfy that requirement we might either allow a and b to also take $-\infty$ and $+\infty$ as values or we might confine a and b within a segment $[A,B]$.

It is somewhat inconvenient that these modifications sacrifice the property that partially inconsistent interval numbers form a group with respect to addition.

Another property which is usually imposed is the existence of Ginsberg involution which is monotonic with respect to $\sqsubseteq$ and antimonotonic with respect to $\leq$. If we consider all partially inconsistent interval numbers without infinities or allow a and b to take $-\infty$ and $+\infty$ values, or if we confine a and b within segment $[-A,A]$, then Ginsberg involution is the weak minus. If we confine a and b within a segment $[A,B]$, then Ginsberg involution maps $[a,b]$ to $[A+B-b,A+B-a]$. One important case here is $[A,B]=[0,1]$.

5 Partially Inconsistent Interval Numbers as a D-frame

For the overview of bitopology, d-frames, and bitopological Stone duality see [4].

Consider the (lower, upper) bitopology on the real line. Define the bilattice isomorphism between d-frame elements, i.e. pairs $\langle L,U\rangle$ of the respective open sets, and partially inconsistent interval numbers. A pair $\langle L,U\rangle$ is a pair of open rays, $\langle(-\infty,a),(b,+\infty)\rangle$ (a and b are allowed to take $-\infty$ and $+\infty$ as values). This pair corresponds to a partially inconsistent interval number $[a,b]$. Consistent, i.e. non-overlapping, pairs of open rays ($a \leq b$) correspond to segments. Total, i.e. covering the whole space, pairs of open rays ($b < a$) correspond to pseudosegments.

6 Bitopological Groups and Antimonotonic Negation

In the context of bitopological groups the following situation is typical: two topologies, T and T^{-1}, are group dual of each other, the multiplication is continuous with respect to both topologies, and the inverse is a bicontinuous map from (X,T,T^{-1}) to its bitopological dual, (X,T^{-1},T) [1].

The minus operation on real numbers is bicontinuous from the (lower, upper) bitoplogy to the (upper, lower) bitopology and vice versa. The corresponding map between the d-frames is very similar to the weak minus (Ginsberg involution), except that the order of bitopological components also needs to be swapped to respect bitopological duality in this case (partially inconsistent interval numbers are a Cartesian product of lower and upper bounds; swapping can be thought of as changing the order of components in this Cartesian product).

In a simular fashion, the true minus operation on the partially inconsistent interval numbers is bicontinuous between a (T,T^{-1}) bitopology on the partially inconsistent interval numbers and its dual (T^{-1},T) bitopology. (Here T and T^{-1} must be group dual topologies of each other, e.g. the Scott topology corresponding to $\sqsubseteq$ and the Scott topology corresponding to $\sqsupseteq$.)

7 Partial and Relaxed Metrics

The standard partial metric on the interval numbers is $p([a_1,b_1],[a_2,b_2]) = \max(b_1,b_2) - \min(a_1,a_2)$. Hence the self-distance for $[a,b]$ is $b-a$. If we extend this formula to pseudosegments, the self-distance of pseudosegments turns out to be negative.

Partial metrics can be understood as upper bounds for "ideal distances". One often has to trade the tightness of those bounds for nicer sets of axioms. E.g. the natural upper bound for the distance between $[0,2]$ and $[1,1]$ is 1, and there is a weak partial metric which yields that. However, if one wants to enjoy the axiom of small self-distances, $p(x,x) \leq p(x,y)$, one has to accept $p([0,2],[1,1]) = 2$, since $p([0,2],[0,2]) = 2$.

A similar trade can be made for lower bounds. The standard interval-valued relaxed metric produces the gap between non-overlapping segments as their lower bound, but takes 0 as the lower bound for the distance between overlapping segments (hence 0 is also the lower bound for self-distance). If one settles for a less tight lower bound and allows the lower bound to be negative in those cases, one can obtain a distance with much nicer properties: $l([a_1,b_1],[a_2,b_2]) = \max(a_1,a_2) - \min(b_1,b_2)$.

We think about the pair $\langle l,p\rangle$ as a relaxed metric valued in partially inconsistent interval numbers. The self-distance of $[a,b]$ is $[a-b,b-a]$ and the self-distance of a pseudosegment is a pseudosegment.

The map $[a,b] \mapsto [b,a]$ expressing the symmetry between segments and pseudosegments also transforms $\langle l,p\rangle$ into $\langle p,l\rangle$.

8 Signed Measures and Signed Multisets

One way to think about $p([a,b],[a,b]) = b - a$ is to say that a pseudosegment has a negative length.

We can also revisit the correspondence between the elements of the (lower, upper) bitopology d-frame, $\{\langle(-\infty,a),(b,+\infty)\rangle\}$, and the partially inconsistent interval numbers. Consider the characteristic function mapping the real line to 1 and subtract from it the characteristic functions of $(-\infty,a)$ and $(b,+\infty)$. If $[a,b]$ is a segment, the result is the characteristic function of that segment (valued 1 for the points belonging to the segment and 0 for the points outside the segment). If $[a,b]$ is a pseudosegment and if we allow for the overlap between $(-\infty,a)$ and $(b,+\infty)$ to be subtracted twice, the result is the generalized characteristic function, which is equal to -1 in the open interval (b,a) and 0 outside (b,a). So we obtain a signed multiset here allowing negative degree of membership.

9 Bilattice-valued Mathematics and Partially Inconsistent Interval Numbers

It seems that mathematics of partial inconsistency should be bilattice-valued.

One recent confirmation of that is a paper by Rodabaugh [6] showing that L-valued bitopology can be understood as L^2-valued topology, and, in particular, that ordinary bitopology can be understood as 4-valued topology. The 4-valued set here is the standard bilattice of 4 elements playing the same role in bitopology as the Sierpinski space plays in topology. The L^2 in general is also a bilattice, with $\sqsubseteq$ being obtained from the product $(L,\sqsubseteq)\times(L,\sqsubseteq)$ and the material order, $\leq$, being obtained from the product of the dual lattice by the original one, $(L,\sqsupseteq)\times(L,\sqsubseteq)$.

All key elements and motives of the partial inconsistency landscape identified by the authors in [2] play interesting roles in mathematics of partially inconsistent interval numbers.

While the fuzzy mathematics in general is lattice-valued, the situations where the lattice is $[0,1]$ or otherwise based on real numbers remain important. Similarly, while mathematics of partial inconsistency is in general likely to be valued in bilattices, the particular situations where the bilattice is based on partially inconsistent real numbers (whether confined within $[0,1]$, $[-1,1]$, or $[-\infty,+\infty]$) are likely to play important roles.

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