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Partial Metrics and Fuzzy Equalities

Joint work with Ralph Kopperman,
Steve Matthews, and Homeira Pajoohesh

BLAST 2009, Las Cruces

Slides for this talk:

http://www.cs.brandeis.edu/∼bukatin/distances_and_equalities.html

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Duality between metric and logical viewpoints

The metric viewpoint: how far two objects are from each other.
The logical viewpoint: to what degree two objects overlap.

Fuzzy mathematics is traditionally done from the logical viewpoint, so the first step in introducing fuzzy metrics is often the transformation f(x,y) = exp(-d(x,y)) (SumTopo2009).

Then we have the following correspondences:

$d(x, y) = 0$ if and only if $f(x, y) = 1$.

$d(x, y) = +\infty$ if and only if $f(x, y) = 0$.

$d(x_1, y_1) < d(x_2, y_2)$ if and only if $f(x_2, y_2) < f(x_1, y_1)$.

The axiom $d(x, x) = 0$ becomes $f(x, x) = 1$.

The axiom $d(x, z) \leq d(x, y) + d(y, z)$ becomes $f(x, y) * f(y, z) \leq f(x, z)$.

Non-expansive maps become maps which respect overlap by not letting it decrease.

Etc..

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$f(x_2, y_2) \sqsubset f(x_1, y_1)$.

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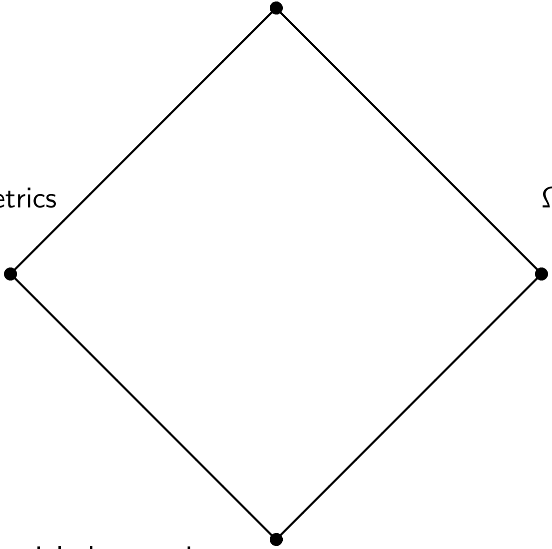
Mathematics of partially defined elements

Generalized distances: instead of $p(x, x) = 0$ axiom,
value $p(x, x)$ expresses how far x is from being completely defined.

Generalized equalities: instead of $x = x$ being always true,
value $=(x, x)$ expresses how well defined x is.

Quantale-valued partial metric = Quantale-valued sets



Partial metrics

Ω-sets

Partial ultrametrics

Example: interval numbers

Consider segments $[a, b]$ and $[c, d]$ on the real line.
Define the distance between them as

$$max(b, d) - min(a, c).$$

partial metrics (Steve Matthews)

$p : X \times X \to \mathbf{R}^+$

$p(x, y) = p(y, x)$ (symmetry)
$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$
$p(x, x) \leq p(x, y)$ (small self-distance)
$p(y, y) + p(x, z) \leq p(x, y) + p(y, z)$ (strong triangularity (Steve Vickers))

Some applications of partial metrics

Good for generalized metrization of a wide class of (usually non-Hausdorff) topologies and bi-topologies.

Main case for computer science so far: the Scott topology.

In the bitopological setting: dually, lower topology, and Lawson topology as join of Scott and lower.

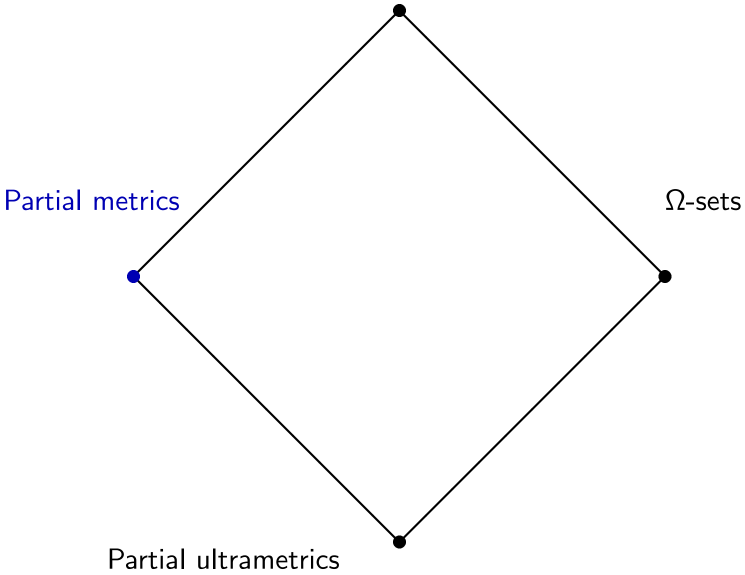
Computer scientists love them, because one can obtain Scott continuous generalized metrizations of Scott domains with partial metrics, when one adopts a logical viewpoint of $\mathbf{R}^+$.

This cannot be done with quasi-metrics, because of $q(x, x) = 0$.

A logical viewpoint of R^+

$\sqsubseteq = \geq$ and we equip this set with a Scott topology.

Quantale-valued partial metric = Quantale-valued sets



Example: partially defined functions

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

The degree of equality of two functions (f, U) and (g, V) is the interior of $\{x \in U \cap V \mid f(x) = g(x)\}$.

Fuzzy equalities valued in complete Heyting algebras

Ω-sets

Ω-valued fuzzy equalities

D.Scott,M.Fourman,D.Higgs (1970s)

complete Heyting algebras

Ω – complete Heyting algebra

complete lattice, $\sqsubseteq$

for all a, b, there is greatest x, denoted as $a \to b$, such that $a \wedge x \sqsubseteq b$.

A topology is a typical complete Heyting algebra: $\sqsubseteq = \subseteq$, $\wedge = \& = \cap$, $U \to V = \mathrm{Int}(V \cup \bar{U})$.

Sets valued in complete Heyting algebras

Ω-valued fuzzy equality: $E : A \times A \to \Omega$

Axioms:

$E(a, b) = E(b, a)$

$E(a, b) \wedge E(b, c) \sqsubseteq E(a, c)$

Example (again): partially defined functions

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

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Another example: pre-sheaves of sets over Ω

Let x be a section over a, and y - a section over b.

$E(x, y) = \bigvee\{p \sqsubseteq a \wedge b \mid x \upharpoonright_p = y \upharpoonright_p\}$.

Distributivity of complete Heyting algebras is used to establish $E(a, b) \wedge E(b, c) \sqsubseteq E(a, c)$.

Some applications: singleton completion and sheafification

Singletons are functions $s : A \to \Omega$, such that

$s(x) \sqsubseteq E(x, x)$

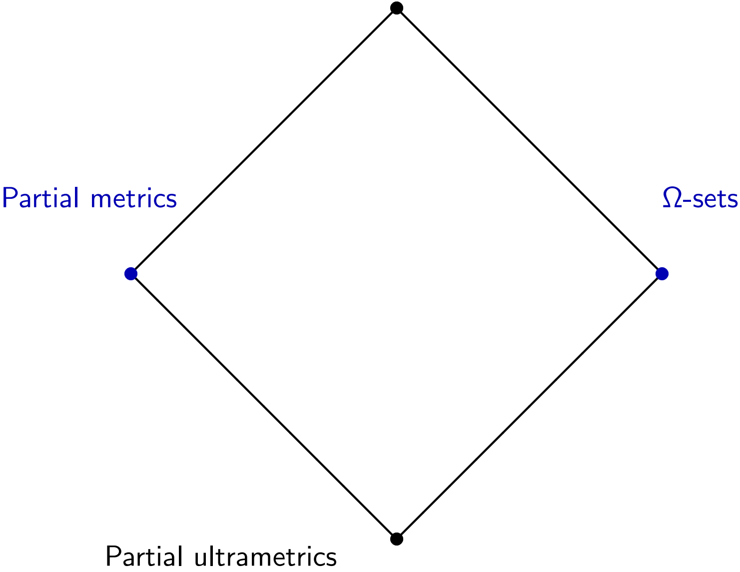
$s(x) \wedge E(x, y) \sqsubseteq s(y)$

$s(x) \wedge s(y) \sqsubseteq E(x, y)$

All functions $y \mapsto E(x, y)$ are singletons. If all singletons are uniquely represented in this way, the Ω-set is called complete. Otherwise, an Ω-set can be mapped into the set of its singletons, resulting in singleton completion.

Complete Ω-sets with morphisms $f : (A, E) \to (B, F)$ satisfying $E(x, x) = F(f(x), f(x))$ and $E(x, y) \sqsubseteq_B F(f(x), f(y))$ are equivalent to the category of sheaves of sets over Ω.

Quantale-valued partial metric = Quantale-valued sets



partial ultrametrics

$p(x, y) = p(y, x)$ (symmetry)
$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$
$p(x, z) \leq max(p(x, y), p(y, z))$

Example: consider set of finite and infite sequences over some alphabet, and denote the length of the common prefix of sequences s_1 and s_2 as $L(s_1, s_2)$. Then define $p(s_1, s_2) = 2^{-L(s_1, s_2)}$.

Partial ultrametrics are partial metrics

$p(x,x) \le max(p(x,y), p(y,x)) = p(x,y)$, obtaining small self-distances.

$p(x,z) \le max(p(x,y), p(y,z))$ means $p(x,z) \le p(x,y)$ or $p(x,z) \le p(y,z)$. Consider $p(x,z) \le p(x,y)$.

We know that $p(y,y) \le p(y,z)$, so $p(x,z) + p(y,y) \le p(x,y) + p(y,z)$, obtaining strong triangularity.

A logical viewpoint of R^+ revisited

$[+\infty, 0]$ can be thought of as the Scott topology on positive reals, $(0, +\infty)$.

x corresponds to the $(x, +\infty)$ Scott open set, so 0 corresponds to the whole space $(0, +\infty)$, and $+\infty$ corresponds to the empty open set.

$\sqsubseteq = \subseteq = \geq$ (order is reversed)

Partial ultrametrics are Ω-sets

Consider $p(x,z) \leq max(p(x,y), p(y,z))$.

Consider $[+\infty, 0]$ as the Scott topology on positive reals.

Then "$p(x,y)$" $\wedge$ "$p(y,z)$" $\sqsubseteq$ "$p(x,z)$".

Or, writing this in details,
$(p(x,y), +\infty) \cap (p(y,z), +\infty) \subseteq (p(x,z), +\infty)$

Partial ultrametrics are Ω-sets

Consider $p(x,z) \leq max(p(x,y), p(y,z))$.

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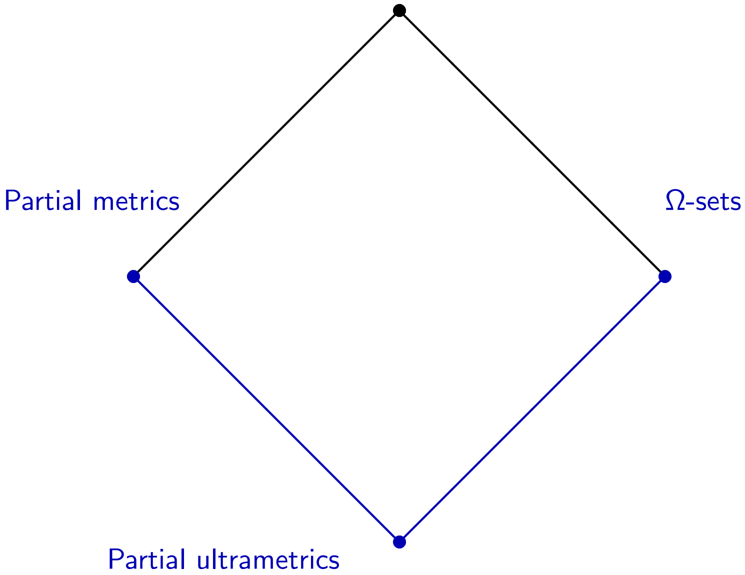
In general, whenever I talk about Ω-sets from a metric viewpoint, and, thus, have $p(x,z) \leq p(x,y) \vee p(y,z)$, I tend to call p a partial ultrametric valued in the system of numbers in question.

Quantale-valued partial metric = Quantale-valued sets



Quantale generalizations

Quantale-valued partial metrics

R.Kopperman, S.Matthews, H.Pajoohesh (2004)

used for generalized metrization of large topologies

Quantale-valued sets

Quantale-valued fuzzy equalities

Ulrich Hoehle (early 1990s)

used for bringing substructal logic to fuzzy sets, e.g. non-idempotent conjunctions such as Lukasiewicz conjunction:

$x \& y = max(0, x + y - 1)$.

Partial pseudometrics into quantales

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

Quantale-valued sets

An M-valued set is a set X equipped with a map $E : X \times X \to M$ (**fuzzy equality**) subject to the axioms

- $E(x,y) \sqsubseteq E(x,x)$
- $E(x,y) = E(y,x)$
- $E(x,y) * (E(y,y) \Rightarrow E(y,z)) \sqsubseteq E(x,z)$

Quantales

Metric viewpoint

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

Logical viewpoint

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Partial pseudometrics into quantales

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Quantales

Metric viewpoint

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

Logical viewpoint

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Do we have a duality between partial metrics and fuzzy equalities?

Specialization orders are the same in both cases (better defined elements are higher).

There are many choices of the notion of morphism, giving various interesting categories, but in all cases we know morphisms go in the same direction yielding equivalences of the categories in question.

So it seems to me that it is more correct to talk about equivalence between partial metrics and fuzzy equalities up to the choice of notation and not about duality between them.

But I am sure that related dualities are lurking nearby.

We noticed the equivalence between partial metrics and fuzzy equalities in 2006:

http://www.cs.brandeis.edu/∼bukatin/distances_and_equalities.html

I want to make a brief pause here

learning from the logical viewpoint

weak semi-cancellativity:

$a \geq b \Rightarrow a = (a \dot{-} b) + b.$

Hoehle imposes its equivalent, and we should also impose it in the theory of partial metrics.

Then we can rewrite the strong triangularity as

$p(y, y) + p(x, z) \leq p(x, y) + p(y, z)$

$E(x, y) * E(y, z) \sqsubseteq E(x, z) * E(y, y)$

(note the "linear logic flavor" of this rule)

Also then the standard theories of equivalences between partial metrics, weighted metrics, and weighted quasi-metrics go through.

learning from the metric viewpoint

Quantale-valued quasi-metrics are from the logical viewpoint not generalized equalities, but generalized partial orders in Lawvere's sense: $E(x,x) \Rightarrow E(x,y)$

They are very useful within the metric viewpoint, and they should be quite useful within the logical viewpoint as well.

bringing metric and logical approaches together

Every separated pre-sheaf of sets over complete Heyting algebra Ω can be thought of as a separated pre-sheaf of Ω-ultrametrics and non-expansive maps.

learning from the metric viewpoint

Base change involves a function from quantale M to quantale N, $\mu : M \to N$, which allows to make N-valued sets from M-valued sets.

In the spirit of Lawvere, Hoehle imposes the rules $\mu(1) = 1$ and $(\mu(x) *_N \mu(y)) \sqsubseteq_N \mu(x *_M y)$.

Our experience of using measures to go from partial ultrametrics valued in algebras of sets to partial metrics valued in reals shows that things work better with an extra rule:
$(\mu(b) \Rightarrow_N \mu(a)) \sqsubseteq_N \mu(b \Rightarrow_M a)$.

Then one can avoid the need to transform to weighted metrics (aka global quantale-valued sets), doing the transform there on weight part and on metric part separately, and then going back to partial metrics (quantale-valued sets).

learning from the logical viewpoint

There are topologically correct partial metrics for Scott domains like $D \cong [D \to D]$.

But we don't know much about the "canonical partial metrics" here, which would "respect this isomorphism" in some sense.

The work I am presenting today is an unexpected side-effect of the efforts to figure this out.

What we learn from the logical viewpoint is that partial metrics valued not in reals, but in the algebras of sets, actually occur in much more pedestrain situations than the need to handle very large topologies.

This tells is that partial (ultra)metrics valued in the algebras of sets actually have computational sense, and then we notice that we do have "canonical" partial ultrametrics valued in the algebras of sets for domains like $D \cong [D \to D]$.

an open problem

The problem to find “good canonical real-valued partial metrics” remains open. The tutorial by Richard Blute actually made me think that if this problem keeps resisting, perhaps it might make sense to look for solutions of $D \cong [D \to D]$ not in Cartesian closed categories like we do now, but in monoidal closed categories.

Perhaps, the category of domains used here should be in harmony with the category of numbers under consideration (we do try to avoid ultrametric solutions here for various reasons).

http://www.cs.brandeis.edu/∼bukatin/distances_and_equalities.html

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