Partial Metrics and Co-continuous Valuations (Extended Version)*

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October 14, 1997

Abstract. The existence of deep connections between partial metrics and valuations is well known in domain theory. However, the treatment of non-algebraic continuous Scott domains have been not quite satisfactory so far.

In this paper we return to the continuous normalized valuations μ on the systems of open sets and introduce notions of *co-continuity* ($\{U_i,\, i \in I\}$ — filtered system of sets $\Rightarrow \mu(\mathrm{Int}(\bigcap_{i\in I} U_i)) = \inf_{i\in I} \mu(U_i)$) and *strong non-degeneracy* ($U \subset V$ — open sets $\Rightarrow \mu(U) < \mu(V)$) for such valuations. We call the resulting class of valuations CC-valuations. The first central result of this paper is a construction of CC-valuations for Scott topologies on all continuous dcpo's with countable bases. This is a surprising result because neither co-continuous, nor strongly non-degenerate valuations are usually possible for ordinary Hausdorff topologies.

Another central result is a new construction of partial metrics. Given a continuous Scott domain A and a CC-valuation μ on the system of the Scott open subsets of A, we construct a continuous partial metric on A yielding the Scott topology as $u(x,y) = \mu(A \setminus (C_x \cap C_y)) - \mu(I_x \cap I_y)$, where $C_x = \{y \in A \mid y \sqsubseteq x\}$ and $I_x = \{y \in A \mid \{x,y\}$ is unbounded$\}$. This construction covers important cases based on real line and allows to obtain an induced metric on $Total(A)$ without unpleasant restrictions known before.

1 Introduction

Lately the theory of *partial metrics* introduced by Matthews [10] undergoes active development and is used in various applications from computational description of metric spaces [5] to the analysis of parallel computations [9]. The

*A preliminary draft. Version 0.5

relationship between partial metrics and *valuations* was first noticed by O'Neill in [11].

In [2] Bukatin and Scott generalized this relationship by considering valuations on powersets of bases, instead of valuations on the domains themselves, as in [11]. They also explained the computational intuition of partial metrics by generalizing them to *relaxed metrics*, which take values in the *interval numbers*.

Partial metrics can be considered as taking values in the *upper bounds* of those interval numbers. However it is often desirable to remove the most restrictive axioms of partial metrics, like *small self-distances*, $u(x,x) \le u(x,y)$, and strong *Vickers-Matthews triangle inequality*, $u(x,z) \le u(x,y) + u(y,z) - u(y,y)$. Thus [2] only requires symmetry and the ordinary triangle inequality for the upper bounds of relaxed metrics.

However, it can be shown (see Section 6) that if the upper bounds $u(x,y)$ of relaxed metrics are based on the idea that common information, or more precisely, *measure of common information* about x and y, brings *negative contribution* to $u(x,y)$ — e.g. in the normalized world we can consider $u(x,y) = 1 - \mu(Info(x) \cap Info(y))$ — then all axioms of partial metrics should hold for u. In fact, it makes sense to introduce both positive and negative information, and to define $u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y))$, then defining meaningful lower bounds $l(x,y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x))$ and obtaining an induced metric on $Total(A)$.

This is, essentially, the approach of the Section 5 of [2], where $Info(x)$ and $Neginfo(x)$ can be understood as subsets of a domain basis. However, there was a number of remaining *open problems*. In particular, while [2] builds partial metrics on all continuous Scott domains with countable bases, the reliance of [2] on the finite weights of non-compact basic elements does not allow to obtain some natural partial metrics on the real-line based domains, and also introduces some unpleasant restrictions on domains which should be satisfied in order to obtain an induced classical metric on $Total(A)$.

1.1 Co-continuous Valuations

This paper rectifies these particular open problems by defining partial metrics via *valuations on the systems of Scott open sets* of domains. The theory of valuations on the open sets underwent a considerable development lately (see [3, 7, 14] and references therein). However we have found that we need a special condition of *co-continuity* for our valuations — for a filtered system of open sets $\{U_i,\ i \in I\}$, $\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I}(\mu(U_i))$. We need this condition to ensure Scott continuity of our partial metrics.

The paper starts as follows. In Section 2 we remind the necessary definitions of domain theory. Section 3 defines various properties of valuations and introduces the class of *CC-valuations* — continuous, normalized, strongly non-degenerate, co-continuous valuations. Section 4 builds a CC-valuation on the system of Scott open sets of every continuous dcpo with a countable basis. This is the first central result of this paper.

It seems that the notion of co-continuity of valuations and this result for the case of continuous Scott domains with countable bases are both new and belong to us. The generalization of this result to continuous dcpo's with countable bases belongs to Klaus Keimel [8]. He worked directly with *completely distributive lattices* of Scott open sets of continuous dcpo's and used the results about completely distributive lattices obtained by Raney in the fifties (see Exercise 2.30 on page 204 of [4]). Here we present a proof which can be considered a simplification of both our original proof and the proof obatined by Keimel. This proof also works for all continuous dcpo's with countable bases. A part of this proof, as predicted by Keimel, can be considered as a special case of Raney's results mentioned above. However, our construction is very simple and self-contained.

Keimel also pointed out that our results are quite surprising, because both co-continuity and *strong non-degeneracy*, $U \subset V$ — open sets $\Rightarrow \mu(U) < \mu(V)$, seem contradictory, as neither of them can hold for the system of open sets of the ordinary Hausdorff topology on $[0,1]$. However, if we replace the system of open sets of this Hausdorff topology with the system of intervals, both conditions would hold. We believe that the reason behind our results is that the Scott topology is coarse enough for its system of open sets to exhibit behaviors similar to the behaviors of typical *bases of open sets* of Hausdorff topologies.

1.2 Application to Partial Metrics

Section 5 discusses partial and relaxed metrics and their properties. Section 6 describes an approach to partial and relaxed metrics where the upper bounds $u(x,y)$ are based on the common information about x and y bringing negative contribution to $u(x,y)$. We formalize this approach introducing the notion of $\mu Info$-structure. However, we feel that this formalization can be further improved.

In particular, Section 6 presents the second central result of this paper — given a CC-valuation on the system of Scott open sets of any continuous Scott domain (no assumptions about the cardinality of the basis are needed here), we build a Scott continuous relaxed metric $< l, u >: A \times A \to \mathbf{R}^I$, such that $u : A \times A \to \mathbf{R}^-$ is a partial metric, the relaxed metric topology coincides with the Scott topology, and if $x, y \in Total(A)$, $l(x,y) = u(x,y)$ and the resulting classical metric $Total(A) \times Total(A) \to \mathbf{R}$ defines a subspace topology on $Total(A)$. Here $\mathbf{R}^I$ is a domain of interval numbers, $\mathbf{R}^-$ — a domain of upper bounds, and $Total(A)$ — a set of maximal elements of A.

Section 7 discusses various examples and possibilities to weaken the strong non-degeneracy condition — to find a sufficiently general weaker condition is an open problem.

Further open problems are discussed in Conclusion.

2 Continuous Scott Domains

Recall that a non-empty partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. A poset, $(A, \sqsubseteq)$, is a *dcpo* if it has the least element, $\perp$, and for any directed $S \subseteq A$, A contains $\sqcup S$ — the least upper bound of S. A set $U \subseteq A$ is *Scott open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$ and for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$. The Scott open subsets of a dcpo form the *Scott topology*.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. $f : A \to B$ is (Scott) continuous iff it is monotonic ($x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$) and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

We define continuous Scott domains in the spirit of [6]. Consider a dcpo $(A, \sqsubseteq)$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$. An element x, such that $x \ll x$, is called *compact*. We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B \in A$.

Consider a set $K \subseteq A$. Assume that $\perp_A \in K$. We say that a dcpo A is a *continuous dcpo* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements. A continuous, bounded complete dcpo is called a *continuous Scott domain*.

3 CC-valuations

Consider a topological space $(X, \mathcal{O})$, where $\mathcal{O}$ consists of all open subsets of X. The following notions of the theory of valuations can be considered standard (for the most available presentation in a regular journal see [3]; the fundamental text in the theory of valuations on Scott opens sets is [7]).

Definition 3.1. A function $\mu : \mathcal{O} \to [0, +\infty]$ is called *valuation* if

1. $\forall U, V \in \mathcal{O}.\ U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
2. $\forall U, V \in \mathcal{O}.\ \mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
3. $\mu(\emptyset) = 0$.

Definition 3.2. A valuation μ is *bounded* if $\mu(X) < +\infty$. A valuation μ is *normalized* if $\mu(X) = 1$.

Remark: If a valuation μ is bounded and $\mu(X) \neq 0$, then it is always easy to replace it with a normalized valuation $\mu'(U) = \mu(U)/\mu(X)$.

Definition 3.3. Define a *directed system of open sets*, $\mathcal{U} = \{U_i, i \in I\}$, as satisfying the following condition: for any finite number of open sets $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_{i_1} \subseteq U_i, \cdots, U_{i_n} \subseteq U_i$.

Definition 3.4. A valuation μ is called *continuous* when for any directed system of open sets $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i)$.

We introduce two new properties of valuations.

Definition 3.5. A valuation $\mu : \mathcal{O} \to [0, +\infty]$ is *strongly non-degenerate* if $\forall U, V \in \mathcal{O}.\ U \subset V \Rightarrow \mu(U) < \mu(V)$.[1]

This is, obviously, a very strong requirement, and we will see later that it might be reasonable to look for weaker non-degeneracy conditions.

Consider a decreasing sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots$, or, more generally, a *filtered system of open sets* $\mathcal{U} = \{U_i, i \in I\}$, meaning that for any finite system of open sets $U_{i_1}, \cdots U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_i \subseteq U_{i_1}, \cdots, U_i \subseteq U_{i_n}$. Consider the interior of the intersection of these sets. It is easy to see that for a valuation μ

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i) \le \inf_{i \in I} \mu(U_i).$$

Definition 3.6. A valuation μ is called *co-continuous* if for any filtered system of open sets $\{U_i,\ i \in I\}$

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i) = \inf_{i \in I} \mu(U_i).$$

Definition 3.7. A continuous, normalized, strong non-degenerate, co-continuous valuation μ is called a *CC-valuation*.

Informally speaking, the strong non-degeneracy provides for non-zero contributions of compact elements and reasonable "pieces of space". The co-continuity provides for single non-compact elements and borders $B \setminus \mathrm{Int}(B)$ of "reasonable" sets $B \subseteq A$ to have zero measures.

"Reasonable" sets here are Alexandrov open (i.e. upwardly closed) sets. Thus, it is possible to consider co-continuity as a method of dealing with non-discreteness of Scott topology. We follow here the remarkable definition of a discrete topology given by Aleksandrov: a topology is discrete if an intersection of arbitrary family of open sets is open (e.g. see [1]). Of course, if one assumes the T_1 separation axiom, then the Aleksandrov's definition implies that all sets are open — the trivial version of the definition, which is unfortunately the only one found in the most textbooks. In this sense, Aleksandrov topology of upwardly closed sets is discrete, but Scott topology is not. The further development of this viewpoint might prove to be fruitful.

We should also notice that since our valuations are bounded, they can be extended onto closed sets via formula $\mu(C) = \mu(A) - \mu(A \setminus C)$, and all definitions of this section can be expressed in the dual form. This gives us considerable benefits and also suggests that the generalization to the case of unbounded valuations is a non-trivial undertaking.

A bounded valuation μ can be uniquely extended to an additive measure defined on the ring of sets generated from the open sets by operations $\cap$, $\cup$, $\setminus$ [12]. We feel that it is useful to draft a possible construction here. We will denote the resulting additive measure also as μ.

[1] We use $U \subset V$ as an equivalent of $U \subseteq V \& U \neq V$.

First, consider all sets $U \cap C$, where U is open and C is closed. They form so-called semiring, which contains $\emptyset$, is closed under intersections, and possesses the following property: If $S = U \cap C$, $S_1 = U_1 \cap C_1$, $S_1 \subset S$, then there is a system of n non-intersecting sets $S_i = U_i \cap C_i$ (in fact, it is enough to take $n = 3$ here), such that $S = \bigcup_{i=1,\ldots,n} S_i$.

Then define $\mu(U \cap C) = \mu(U) - \mu(U \setminus C)$, using the fact that $U \setminus C = U \cap \overline{C}$ is an open set. Then observe that this definition is correct, i.e. if $U \cap C = U_1 \cap C_1$ then $\mu(U) - \mu(U \setminus C) = \mu(U_1) - \mu(U_1 \setminus C_1)$, and that the resulting measure is additive for the decomposition above: $\mu(S) = \mu(S_1) + \ldots + \mu(S_n)$. Then the unique extension to the ring of sets in question is the standard result of measure theory.

The issues of σ-additivity are not in the scope of this text (interested readers are referred to [7]). We deal with the specific infinite systems of sets we need, and mainly focus on quite orthogonal conditions given to us by co-continuity of μ.

3.1 Example: Valuations Based on Weights of Basic Elements

This example essentially reproduces a construction in [2]. Consider a continuous dcpo A with a countable basis K. Assign the converging system of weights to basic elements: $w(k) > 0$, $\sum_{k \in K} w(k) = 1$. Define $\mu(U) = \sum_{k \in U} w(k)$. It is easy to see that μ is a continuous, normalized, strongly non-degenerate valuation.

However, μ is co-continuous if and only if all basic elements are compact (which is possible only if A is algebraic). To see why this is so, we undertake a small discourse, which will be useful for us later in this paper.

First, observe that intersection of arbitrary system of Alexandrov open (i.e. upwardly closed) sets is Alexandrov open. Also it is a well-known fact that $\{y \mid y \succ\succ x\}$ is Scott open in a continuous dcpo.

Lemma 3.1 (Border Lemma) *Consider an Alexandrov open set $B \subseteq A$. Then its interior in the Scott topology, $\mathrm{Int}(B) = \{y \in A \mid \exists x \in B.\, x \prec\prec y\}$. Correspondingly, the border of B in the Scott topology, $B \setminus \mathrm{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\, x \prec\prec y)\}$*

Lemma 3.2 *If K is a basis of compact elements, then $\mu(U) = \sum_{k \in U} w(k)$, where $w(k) > 0$, $\sum_{k \in K} w(k) = 1$, defines a CC-valuation.*

Proof. (of co-continuity) Consider a filtered system of open sets $\{U_i,\, i \in I\}$. The set $B = \bigcap_{i \in I} U_i$ is Alexandrov open. Because $\{U_i\}$ is filtered, it is easy to see that $\sum_{k \in B} w(k) = \inf_{i \in I} \mu(U_i)$. Because basic elements of K are compact, $k \in K$, $k \in B \Rightarrow k \in \mathrm{Int}(B)$. Hence $\sum_{k \in B} w(k) = \sum_{k \in \mathrm{Int}(B)} w(k)$.

$\square$

Lemma 3.3 *If K contains a non-compact element x, then $\mu(U) = \sum_{k \in U} w(k)$, where $w(k) > 0$, $\sum_{k \in K} w(k) = 1$ is not co-continuous.*

Proof. By the definition of basis, $x = \sqcup K_x$, where $K_x = \{k \in K \mid k \prec\!\!\prec x\}$ is a directed set. For any k, consider $U_k = \{y \mid y \succ\!\!\succ k\}$. Because K_x is directed, it is easy to see that $\{U_k,\, k \in K_x\}$ is filtered. Consider $B = \bigcap_{k \in K_x} U_k$. It is easy to see that $x \in B$.

Let us show that $x \notin \mathrm{Int}(B)$. If $x \in \mathrm{Int}(B)$ then there is $y \in B$ such that $y \prec\!\!\prec x$ by the Border Lemma. Then $y \sqsubseteq k$ for some $k \in K_x$, and since B is upwardly closed, $k \in B$. This means that $\forall k' \in K_x.\ k' \prec\!\!\prec k$ yielding $x \sqsubseteq k$ and $k \prec\!\!\prec x$. Therefore $k = x$ and x is compact, which is a contradiction.

Thus,

$$x \notin \mathrm{Int}(B) \Rightarrow \sum_{k \in B} w(k) \geq \sum_{k \in \mathrm{Int}(B)} w(k) + w(x) > \sum_{k \in \mathrm{Int}(B)} w(k)$$

Since $\sum_{k \in B} w(k) = \inf_{k \in K_x} \mu(U_k)$ the Lemma is proved.

□

3.2 A Vertical Segment of Real Line

Consider the segment $[0,\, 1]$, $\sqsubseteq = \leq$. Define $\mu((x,\, 1]) = 1 - x$. Unfortunately, to ensure strong non-degeneracy we have to define $\mu([0,\, 1]) = 1 + \epsilon$, $\epsilon > 0$. This is the first hint that strong non-degeneracy might constitute too strong restriction in many cases. In order to obtain a normalized valuation we have to consider $\mu'(U) = \mu(U)/(1+\epsilon)$. The resulting μ' is a CC-valuation.

We will consider more sophisticated examples, involving interval numbers, later on.

4 Constructing CC-valuations

In this section we build a CC-valuation for all continuous dcpo's with countable bases. The construction generalizes the one of Subsection 3.1. We are still going to assign weights, $w(k) > 0$, to compact elements. For non-compact basic elements we proceed as follows. We focus our attention on the pairs of non-compact basic elements, (k', k''), which do not have any compact elements between them, and call such elements *continuously connected*. We observe, that for every such pair we can construct a special kind of vertical chain, which "behaves like a vertical segment $[0,1]$ of real line". We call such chain a *stick*. We assign weights, $v(k', k'') > 0$, to sticks as well, in such a way that the sum of all $w(k)$ and all $v(k', k'')$ is 1.

As in Subsection 3.1, compact elements k contribute $w(k)$ to $\mu(U)$, if $k \in U$. An intersection of the stick, associated with a continuously connected pair

(k', k''), with an open set U "behaves as either $(q, 1]$ or $[q, 1]$", where $q \in [0, 1]$. Such stick contributes $(1 - q) \cdot v(k', k'')$ to $\mu(U)$. The resulting μ is the desired CC-valuation.

It is possible to think, that a complete lattice homomorphism from the lattice of Scott open sets to $[0, 1]$ is associated with every compact element and with every stick defined by basic continuously connected elements, k' and k''. Then, as suggested by Keimel [8], all these homomorphisms together can be thought of as an injective complete lattice homomorphism to $[0, 1]^J$. From this point of view, our construction of μ is the same as in [8].

Thus the discourse in this section yields the proof of the following:

Theorem 4.1 *For any continuous dcpo A with a countable basis, there is a CC-valuation μ on the system of its Scott open sets.*

4.1 Continuous Connectivity and Sticks

Definition 4.1. Two elements $x \prec\!\prec y$ are called *continuously connected* if the set $\{k \in A | k - \text{compact},\ x \prec\!\prec k \prec\!\prec y\}$ is empty.

Remark: This implies that x and y are not compact.

Lemma 4.1 *If $x \prec\!\prec y$ are continuously connected, then $\{z \,|\, x \prec\!\prec z \prec\!\prec y\}$ has cardinality of at least continuum.*

Proof. We use the well-known theorem on intermediate values that $x \prec\!\prec y \Rightarrow \exists z \in A\ x \prec\!\prec z \prec\!\prec y$ (see [6]).Applying this theorem again and again we build a countable system of elements between x and y as follows, using rational numbers as indices for intermediate elements:

$$x \prec\!\prec a_{1/2} \prec\!\prec y, \quad x \prec\!\prec a_{1/4} \prec\!\prec a_{1/2} \prec\!\prec a_{3/4} \prec\!\prec y, \ldots$$

All these elements are non-compact and hence non-equal. Now consider a directed set $\{a_i \,|\, i \leq r\}$, where r is a real number, $0 < r < 1$. Introduce $b_r = \sqcup\{a_i \,|\, i \leq r\}$. We prove that if $r < s$ then $b_r \prec\!\prec b_s$, and also that $x \prec\!\prec b_r \prec\!\prec b_s \prec\!\prec y$, thus obtaining the required cardinality. Indeed it is easy to find such n and numbers q_1, q_2, q_3, q_4, that

$$x \prec\!\prec a_{q_1/2^n} \sqsubseteq b_r \sqsubseteq a_{q_2/2^n} \prec\!\prec a_{q_3/2^n} \sqsubseteq b_s \prec\!\prec a_{q_4/2^n} \prec\!\prec y$$

□

Definition 4.2. We call the set of continuum different non-compact elements $\{a_r \,|\, r \in (0, 1)\}$ between continuously connected $x \prec\!\prec y$, built in the proof above, such that $x \prec\!\prec a_r \prec\!\prec a_q \prec\!\prec z \Leftrightarrow r < q$ a (vertical) *stick*.

4.2 Proof of Theorem 4.1

Consider a continuous dcpo A with a countable basis K. As discussed earlier, with every compact $k \in K$ we associate weight $w(k) > 0$, and with every continuously connected pair (k', k''), $k', k'' \in K$, we associate weight $v(k', k'') > 0$ and a stick $\{a_r^{k',k''} \mid r \in (0,1)\}$. Since K is countable, we can require $\sum w(k) + \sum v(k', k'') = 1$.

Whenever we have an upwardly closed (i.e. Alexandrov open) set U, for any stick $\{a_r^{k',k''} \mid r \in (0,1)\}$ there is a number $q_U^{k',k''} \in [0,1]$, such that $r < q_U^{k',k''} \Rightarrow a_r^{k',k''} \notin U$ and $q_U^{k',k''} < r \Rightarrow a_r^{k',k''} \in U$. In particular, for a Scott open set U define

$$\mu(U) = \sum_{k-\text{compact}, k \in U} w(k) + \sum_{k',k'' \in K-\text{continuously connected}} (1 - q_U^{k',k''}) \cdot v(k', k'')$$

It is easy to show that μ is a normalized valuation. The rest follows from the following Lemmas.

Lemma 4.2 *μ is continuous.*

Proof. Consider a directed system of open sets, $\{U_i, i \in I\}$, and $U = \bigcup_{i \in I} U_i$. We need to show that for any $\epsilon > 0$, there is such $U_i, i \in I$, that $\mu(U) - \mu(U_i) < \epsilon$.

Take enough (a finite number) of compact elements, $k_1, \ldots, k_n$, and continuously connected pairs of basic elements, $(k'_1, k''_1), \ldots, (k'_m, k''_m)$, so that $w(k_1) + \ldots + w(k_n) + v(k'_1, k''_1) + \ldots + v(k'_m, k''_m) > 1 - \epsilon/2$. For each $k_j \in U$, take $U_{i_j}, i_j \in I$, such that $k_j \in U_{i_j}$. For each (k'_j, k''_j), such that $q_U^{k'_j,k''_j} < 1$, take $U_{i'_j}, i'_j \in I$, such that $q_{U_{i'_j}}^{k'_j,k''_j} - q_U^{k'_j,k''_j} < \epsilon/2$. An upper bound of these U_{i_j} and $U_{i'_j}$ is the desired U_i.

□

Lemma 4.3 *μ is strongly non-degenerate.*

Proof. Let U and V be Scott open subsets of A and $U \subset V$. Let us prove that $V \setminus U$ contains either a compact element or a stick between basic elements. Take $x \in V \setminus U$. If x is compact, then we are fine. Assume that x is not compact. We know that $x = \sqcup K_x$, $K_x = \{k \in K \mid k \prec\prec x\}$ is directed set. Since V is open $\exists k \in K_x.\ k \in V$. Since $k \sqsubseteq x$ and $x \notin U$, $k \in V \setminus U$. If there is k' – compact, such that $k \prec\prec k' \prec\prec x$, we are fine, since $k' \in V \setminus U$. Otherwise, since any basis includes all compact elements, k and x are continuously connected.

Now, as in the theorem of intermediate values $x = \sqcup \widetilde{K}_x$, $\widetilde{K}_x = \{k' \in K \mid \exists k'' \in K.\ k' \prec\prec k'' \prec\prec x\}$ is directed set, thus $\exists k'\, k''.\ k \sqsubseteq k' \prec\prec k'' \prec\prec x$, thus (k, k'') yields the desired stick.

If $k \in V \setminus U$ and k is compact, then $\mu(V) - \mu(U) \geq w(k) > 0$. If the stick formed by (k, k') is in $V \setminus U$, then $\mu(V) - \mu(U) \geq v(k, k') > 0$.

□

Lemma 4.4 *μ is co-continuous.*

Proof. Recall the development in Subsection 3.1. Consider a filtered system of open sets $\{U_i,\ i \in I\}$. By Lemma 3.1 for $B = \bigcap_{i\in I} U_i$, $B \setminus \mathrm{Int}(B) = \{y \in B \,|\, \neg(\exists x \in B.\ x \prec\!\prec y)\}$. Notice that $B \setminus \mathrm{Int}(B)$, in particular, does not contain compact elements. Another important point is that for any stick, $q_B^{k',k''} = q_{\mathrm{Int}(B)}^{k',k''}$.

The further development is essentially dual to the proof of Lemma 4.2. We need to show that for any $\epsilon > 0$, there is such U_i, $i \in I$, that $\mu(U_i) - \mu(\mathrm{Int}(B)) < \epsilon$.

Take enough (a finite number) of compact elements, $k_1, \ldots, k_n$, and continuously connected pairs of basic elements, $(k'_1, k''_1), \ldots, (k'_m, k''_m)$, so that $w(k_1) + \ldots + w(k_n) + v(k'_1, k''_1) + \ldots + v(k'_m, k''_m) > 1 - \epsilon/2$. For each $k_j \notin \mathrm{Int}(B)$, take U_{i_j}, $i_j \in I$, such that $k_j \notin U_{i_j}$. For each (k'_j, k''_j), such that $q_{\mathrm{Int}(B)}^{k'_j,k''_j} > 0$, take $U_{i'_j}$, $i'_j \in I$, such that $q_{\mathrm{Int}(B)}^{k'_j,k''_j} - q_{U_{i'_j}}^{k'_j,k''_j} < \epsilon/2$. A lower bound of these U_{i_j} and $U_{i'_j}$ is the desired U_i.

□

5 Partial and Relaxed Metrics on Domains

The motivations behind the notion of relaxed metric, its computational meaning and its relationships with the partial metrics [10] were explained in [2]. Here we focus on the definitions and basic properties, revisit the issue of specific axioms of partial metrics, and mention the relevant open problems.

The distance domain consists of pairs $< a, b >$ (also denoted as $[a, b]$) of non-negative reals ($+\infty$ included), such that $a \leq b$. We denote this domain as R^I. $[a, b] \sqsubseteq_{R^I} [c, d]$ iff $a \leq c$ and $d \leq b$.

We can also think about R^I as a subset of $R^+ \times R^-$, where $\sqsubseteq_{R^+} = \leq$, $\sqsubseteq_{R^-} = \geq$, and both R^+ and R^- consist of non-negative reals and $+\infty$. We call R^+ a *domain of lower bounds*, and R^- — a *domain of upper bounds*. Thus a distance function $\rho : A \times A \to R^I$ can be thought of as a pair of distance functions $< l, u >$, $l : A \times A \to R^+$, $u : A \times A \to R^-$.

Definition 5.1. A symmetric function $u : A \times A \to R^-$ is called a *relaxed metric* when it satisfies the triangle inequality. A symmetric function $\rho : A \times A \to R^I$ is called a *relaxed metric* when its upper part u is a relaxed metric.

An *open ball* with a center $x \in A$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x, y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those points y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

Definition 5.2. A subset U of A is *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > u(x,x)$ such that $B_{x,\epsilon} \subseteq U$.

It is easy to show that for a continuous relaxed metric on a dcpo all relaxed metric open sets are Scott open and form a topology.

The distances p with $p(x,x) \neq 0$ were first introduced by Matthews [10, 9]. They are known as *partial metrics* and obey the following axioms:

1. $x = y$ iff $p(x,x) = p(x,y) = p(y,y)$.
2. $p(x,x) \leq p(x,y)$.
3. $p(x,y) = p(y,x)$.
4. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Whenever partial metrics are used to describe a partially ordered domain, a stronger form of the first two axioms is used: If $x \sqsubseteq y$ then $p(x,x) = p(x,y)$, otherwise $p(x,x) < p(x,y)$. We include the stronger form in the definition of partial metrics for the purposes of this paper.

5.1 Partial Metrics and Functoriality

The Section 8.1 of [2] discusses the issue of whether axioms $u(x,x) \leq u(x,y)$ and $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$ should hold for the upper bounds of relaxed metrics. In particular, the approach in this paper is based on $u(x,y) = 1 - \mu(\text{Common information between } x \text{ and } y)$ and thus, as will be explained in details in the next section, axioms of partial metrics hold.

Here I would like to emphasize the utilitarian value of these axioms, i.e. certain useful properties which are easy to establish in the presence of these axioms. In all cases these considerations are closely related to the following series of *open problems*: What are the necessary and/or sufficient conditions for relaxed/partial metrics on various classes of domains in order for all Scott open sets to be relaxed metric open?

5.1.1 Open Balls Should be Open Sets

The strong Vickers–Matthews triangle inequality, $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$, is helpful in the proof that an open ball is a relaxed open set.

The proof goes as follows. Consider $y \in B_{x,\epsilon}$. We need to find $\delta > 0$, such that $B_{y,\delta+u(y,y)} \subseteq B_{x,\epsilon}$, i.e. $\forall z.\ u(y,z) \leq u(y,y) + \delta \Rightarrow u(x,z) < \epsilon$. Using $u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$ and $\epsilon - u(x,y) > 0$ we take $\delta = (\epsilon - u(x,y))/2$ and obtain the desired result. This is closely related to the fact noted by Vickers, that for $d(x,y) = u(x,y) - u(x,x)$, $d(x,z) \leq d(x,y) +$

$d(y,z) \Leftrightarrow u(x,z) \leq u(x,y) + u(y,z) - u(y,y)$. The ordinary triangle inequality $u(x,z) \leq u(x,y) + u(y,z)$ is not enough here, because we are looking for the ball around y with the radius $u(y,y) + \delta$ and not just δ.

5.1.2 Functoriality

One of the open issues in the field remains to develop a *functorial* approach to relaxed metrics. This approach should facilitate our abilities to both define canonical distances on domains and to compute these distances at a reasonable cost.

This method is to define reasonable relaxed metrics on some specific elementary domains, and then, given reasonable structures (relaxed metrics, or sometimes, measures) on domains A, B, define relaxed metric on domains $A \times B$, $A \to B$, etc. The natural candidates for relaxed metric on $A \times B$ and $A \to B$ are

$$\rho_{A\times B}(< x, y >, < x', y' >) = \alpha \cdot \rho_A(x) + \beta \cdot \rho_B(y),\ \alpha,\, \beta > 0,\ \alpha + \beta = 1$$

$$\rho_{A\to B}(f,g) = \int_{x\in A} \rho_B(f(x), g(x)) d\mu_x.$$

We also hope to be able to extend this approach to reflexive domains and to obtain some invariance properties of ρ in the process.

In all cases we would like to ensure that relaxed metric topology and Scott topology coincide. In particular, in order for Scott open sets to be relaxed metric open it is *necessary* (not sufficient) that $u(x,x) \geq u(x,y) \Rightarrow x \sqsubseteq y$.

However, for general relaxed metrics it is easy to come up with examples, when for both A and B $u_A(x,x) \geq u_B(x,y) \Rightarrow x \sqsubseteq y$ and $u_B(x',x') \geq u_B(x',y') \Rightarrow x' \sqsubseteq y'$, but $u_{A\times B}(< x, x' >, < x, x' >) \geq u_{A\times B}(< x, x' >, < y, y' >) \not\Rightarrow (x \sqsubseteq y \& x' \sqsubseteq y')$, and the similar situation takes place for $A \to B$. However, the strong form of the axiom of small self-distances, $u(x,x) \leq u(x,y)$ (namely, $x \sqsubseteq y \Rightarrow u(x,x) = u(x,y)$, $x \not\sqsubseteq y \Rightarrow u(x,x) < u(x,y)$) seems to eliminate these problems.

6 Partial and Relaxed Metrics via Information

6.1 $\mu Info$-structures

Some of the earlier known constructions of partial metrics can be understood via the mechanism of *common information* between elements x and y bringing negative contribution to $u(x,y)$ (see [2, Section 8]). This can be further formalized as follows. Assume that there is a set $\mathcal{I}$ representing information about elements of a dcpo A. We choose a ring, $\mathcal{M}(\mathcal{I})$, of admissible subsets of $\mathcal{I}$ and introduce a measure-like structure, μ, on $\mathcal{M}(\mathcal{I})$. We associate a set, $Info(x) \in \mathcal{M}(\mathcal{I})$, with every $x \in A$, and call $Info(x)$ a set of (positive) information about x. We also would like to consider negative information about x, $Neginfo(x) \in \mathcal{M}(\mathcal{I})$, — intuitively speaking, this is information which cannot become true about x, when x is arbitrarily increased.

Definition 6.1. Given a dcpo A, the tuple of $(A, \mathcal{I}, \mathcal{M}(\mathcal{I}), \mu, Info, Neginfo)$ is called a *$\mu Info$-structure* on A, if $\mathcal{M}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$ — a ring of subsets closed with respect to $\cap$, $\cup$, $\setminus$ and including $\emptyset$ and $\mathcal{I}$, $\mu : \mathcal{M}(\mathcal{I}) \to [0, 1]$, $Info : A \to \mathcal{M}(\mathcal{I})$, and $Neginfo : A \to \mathcal{M}(\mathcal{I})$, and the following axioms are satisfied:

1. **(VALUATION AXIOMS)**
   (a) $\mu(\mathcal{I}) = 1, \ \mu(\emptyset) = 0$;
   (b) $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
   (c) $\mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;

2. **($Info$ AXIOMS)**
   (a) $x \sqsubseteq y \Leftrightarrow Info(x) \subseteq Info(y)$;
   (b) $x \sqsubset y \Rightarrow Info(x) \subset Info(y)$;

3. **($Neginfo$ AXIOMS)**
   (a) $Info(x) \cap Neginfo(x) = \emptyset$;
   (b) $x \sqsubseteq y \Rightarrow Neginfo(x) \subseteq Neginfo(y)$;

4. **(STRONG RESPECT FOR TOTALITY)**
   $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$;

5. **(CONTINUITY OF INDUCED RELAXED METRIC)**
   if B is a directed subset of A and $y \in A$, then
   (a) $\mu(Info(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Info(x) \cap Info(y))$,
   (b) $\mu(Info(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Info(x) \cap Neginfo(y))$,
   (c) $\mu(Neginfo(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Info(y))$,
   (d) $\mu(Neginfo(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Neginfo(y))$;

6. **(SCOTT OPEN SETS ARE RELAXED METRIC OPEN)**
   for any (basic) Scott open set $U \subseteq A$ and $x \in U$, there is an $\epsilon > 0$, such that $\forall y \in A.\ \mu(Info(x)) - \mu(Info(x) \cap Info(y)) < \epsilon \Rightarrow y \in U$.

In terms of lattice theory, μ is a (normalized) valuation on a lattice $\mathcal{M}(\mathcal{I})$. The consideration of unbounded measures is beyond the scope of this paper, and $\mu(\mathcal{I}) = 1$ is assumed for convenience. Axioms relating $\sqsubseteq$ and $Info$ are in the spirit of information systems approach [13], although we are not considering any inference structure over $\mathcal{I}$ in this paper.

The requirements for negative information are relatively weak, because it is quite natural to have $\forall x \in A.\ Neginfo(x) = \emptyset$ if A has the top element.

The axiom that for $x \in Total(A)$, $Info(x) \cup Neginfo(x) = \mathcal{I}$, is desirable because indeed, if some $i \in \mathcal{I}$ does not belong to $Info(x)$ and x can not be further increased, then by our intuition behind $Neginfo(x)$, i should belong to $Neginfo(x)$. However, this axiom might be too strong and will be further discussed later.

The last two axioms, strictly speaking, represent "cheating" on our side — they almost immediately imply the properties, after which they are named, but they are complicated and might be difficult to establish. We hope, that these axioms will be replaced by something more tractable in the future. One of the obstacles seems to be the fact in some valuable approaches (in particular, in this paper) it is not correct that $x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ implies that $Info(\sqcup_{i \in \mathbf{N}} x_i) = \bigcup_{i \in \mathbf{N}} Info(x_i)$.

The nature of these set-theoretical representations, $\mathcal{I}$, of domains may vary: one can consider sets of tokens of information systems, powersets of domains bases, or powersets of domains themselves, custom-made sets for specific domains, etc. The approach via powersets of domain bases (see [2]) can be thought of as a partial case of the approach via powersets of domains themselves adopted in the present paper.

6.2 Partial and Relaxed Metrics via $\mu Info$-structures

Define the (upper estimate of the) distance between x and y from A as $u : A \times A \to \mathbf{R}^-$:

$$u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y)).$$

I.e. the more information x and y have in common the smaller is the distance between them. However a partially defined element might not have too much information at all, so its self-distance $u(x,x) = 1 - \mu(Info(x)) - \mu(Neginfo(x))$ might be large.

It is possible to find information which will never made it into $Info(x) \cap Info(y)$ or $Neginfo(x) \cap Neginfo(y)$ even when x and y are arbitrarily increased. In particular, $Info(x) \cap Neginfo(y)$ and $Info(y) \cap Neginfo(x)$ represent such information. Then we can introduce the lower estimate of the distance $l : A \times A \to \mathbf{R}^+$:

$$l(x,y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x)).$$

The proof of Lemma 9 of [2] is directly applicable and yields $(Info(x) \cap Neginfo(y)) \cup (Info(y) \cap Neginfo(x)) \subseteq Info \setminus ((Info(x) \cap Info(y)) \cup (Neginfo(x) \cap Neginfo(y)))$ implying $l(x,y) \leq u(x,y)$. Thus we can form an **induced relaxed metric**, $\rho : A \times A \to R^I$, $\rho = \, < l, u >$, with a meaningful lower bound.

Now certain properties of partial and relaxed metrics can be established.

1. $u(x,x) \le u(x,y)$.
   Indeed, axiom $U \subseteq V \Rightarrow \mu(U) \le \mu(V)$ implies $\mu(Info(x) \cap Info(y)) \le \mu(Info(x))$, $\mu(Neginfo(x) \cap Neginfo(y)) \le \mu(Neginfo(x))$, yielding $u(x,x) \le u(x,y)$.

2. $u(x,z) \le u(x,y) + u(y,z) - u(y,y)$.
   The proof of Lemma 11 of [2] goes through.

   The symmetry $u(x,y) = u(y,x)$ is obvious. Another helpful fact is

3. $x \sqsubseteq y \Rightarrow u(x,x) = u(x,y)$.
   We are using the axiom $x \sqsubseteq y \Rightarrow Info(x) \subseteq Info(y)$, which implies $Info(x) \cap Info(y) = Info(x)$, and the same for $Neginfo$.

4. $x \not\sqsubseteq y \Rightarrow u(x,x) < u(x,y)$.

   This follows from the following fact:

5. Any Scott open set is relaxed metric open.

   This, in turn, follows at once from the last axiom of $\mu Info$-structure and $\mu(Neginfo(x) \cap Neginfo(y)) \le \mu(Neginfo(x))$.

6. Any relaxed metric open set is Scott open.

   This follows at once from the following property:

7. The induced relaxed metric is a Scott continuous function.

   This also follows immediately from the corresponding axiom of $\mu Info$-structure.

Thus, the following theorem is proved (observe, that the **strong respect for totality** axiom is NOT used in the proof).

Theorem 6.1 *The function u is a partial metric. The function ρ is a continuous relaxed metric. The relaxed metric topology coincides with the Scott topology.*

Due to the axiom $\forall x \in Total(A).\ Info(x) \cup Neginfo(x) = \mathcal{I}$, the proof of Lemma 10 of [2] would go through, yielding

$$x, y \in Total(A) \Rightarrow l(x,y) = u(x,y)$$

and allowing to obtain the following theorem (cf. Theorem 8 of [2]).

Theorem 6.2 *For all x and y from $Total(A)$, $l(x,y) = u(x,y)$. Consider $\mu : Total(A) \times Total(A) \to \mathbf{R}$, $\mu(x,y) = l(x,y) = u(x,y)$. Then $(Total(A), \mu)$ is a metric space, and its metric topology is the subspace topology induced by the Scott topology on A.*

However, in [2] $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ holds under an awkward condition of the regularity of the basis. While bases of algebraic Scott domains and of continuous lattices can be made regular, there are important continuous Scott domains, which cannot be given regular bases. In particular, in $\mathbf{R}^I$ no element, except for $\perp$, satisfies the condition of regularity, hence a regular basis cannot be provided for $\mathbf{R}^I$.

The achievement of the construction to be described in the Section 6.4 is that by removing the reliance on the weights of non-compact basic elements, it eliminates the regularity requirement and implies $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ for all continuous Scott domains equipped with a CC-valuation (which is built above for all continuous Scott domains with countable bases) where $Info(x)$ and $Neginfo(x)$ are as described below in the Subsection 6.4.

However, it still might be fruitful to consider replacing the axiom $\forall x \in Total(A).\ Info(x) \cup Neginfo(x) = \mathcal{I}$ by something like $\forall x \in Total(A).\ \mu(\mathcal{I} \setminus (Info(x) \cup Neginfo(x))) = 0$.

6.3 A Previously Known Construction

Here we recall a construction from [2] based on a generally non-co-continuous valuation of Subsection 3.1. We will reformulate it in our terms of $\mu Info$-structures. In [2] it was natural to think that $\mathcal{I} = K$. Here we reformulate that construction in terms of $\mathcal{I} = A$, thus abandoning the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ altogether.

Define $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$, $P_x = \{y \in A \mid y \prec\!\prec x\}$ (cf. $I_x = \{k \in K \mid \{k, x\} \text{ is unbounded}\}$, $K_x = \{k \in K \mid k \prec\!\prec x\}$ in [2]).

Define $Info(x) = P_x$, $Neginfo(x) = I_x$. Consider a valuation μ of Subsection 3.1: for any $S \subset \mathcal{I} = A$, $\mu(S) = \sum_{k \in S \cap K} w(k)$. μ is a continuous strongly non-degenerate valuation, but it is not co-continuous unless K consists only of compact elements.

Because of this we cannot replace an inconvenient definition of $Info(x) = P_x$ by $Info(x) = C_x = \{y \in A \mid y \sqsubseteq x\}$ (which would restore the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = A$) as $\mu(C_k)$ would not be equal to $\sup_{k' \prec\prec k} \mu(C_{k'})$ if k is a non-compact basic element, leading to the non-continuity of the partial metric $u(x, y)$.

Also the reliance on countable system of finite weights excludes such natural partial metrics as metric $u : \mathbf{R}^-_{[0,1]} \times \mathbf{R}^-_{[0,1]} \to \mathbf{R}^-$, where $\mathbf{R}^-_{[0,1]}$ is the set $[0, 1]$ equipped with the dual partial order $\sqsubseteq = \geq$, and $u(x, y) = max(x, y)$. We rectify all these problems in the next Subsection.

6.4 Partial and Relaxed Metrics via CC-valuations

Assume that there is a CC-valuation $\mu(U)$ on Scott open sets of a domain A. Then it uniquely extends to an additive measure μ on the ring of sets generated by the system of open sets. Define $\mathcal{I} = A$, $Info(x) = C_x$, $Neginfo(x) =$

I_x. It is easy to see that **valuation**, $Info$, and $Neginfo$ axioms of $\mu Info$-structure hold. We have $x \in Total(A) \Rightarrow C_x \cup I_x = A$. Thus we only need to establish $\mu Info$-structure axioms of **continuity of induced relaxed metrics** and **Scott open sets are relaxed metric open** in order to prove theorems 6.1 and 6.2 for our induced relaxed metric ($u(x,y) = 1 - \mu(C_x \cap C_y) - \mu(I_x \cap I_y)$, $l(x,y) = \mu(C_x \cap I_y) + \mu(C_y \cap I_x)$). This axioms are estabslished by the Lemmas below.

You will also see that for such bare-bones partial metrics, as $u(x,y) = 1 - \mu(C_x \cap C_y)$, which are nevertheless quite sufficient for topological purposes and for domains with $\top$, only *co-continuity* matters, continuity is not important.

Observe also that since the construction in Section 3.1 does form a CC-valuation for algebraic Scott domains with bases of compact elements, the construction in [2] can be considered as a partial case of our current construction if the basis does not contain non-compact elements.

Lemma 6.1 *Assume that μ is a co-continuous valuation and B is a directed subset of A. Then $\mu(C_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(C_x \cap Q))$, where Q is a closed or open subset of A.*

Remark: Note that continuity of μ is not required here.

Proof. Part A: Let Q be a closed subset of A. Then $(C_x \cap Q,\ x \in B)$ form a directed set of closed sets. We need to show that $\overline{(\bigcup_{x \in B} C_x) \cap Q} = C_{\sqcup B} \cap Q$, then the result will follow by co-continuity. "$\subseteq$" is trivial. Let us proof "$\supseteq$". Consider $x \in C_{\sqcup B} \cap Q$. $x = \sqcup P_x$; $P_x = \{y \in A \mid y \prec\!\prec x\}$ is directed set and $P_x \subseteq C_{\sqcup B} \cap Q$. However, for $\forall y \in P_x$, since $y \prec\!\prec x$ and $x \sqsubseteq \sqcup B$, $\exists z \in B.\ y \sqsubseteq z$, hence $y \in C_z \cap Q$. Hence $P_x \subseteq \bigcup_{z \in B}(C_z \cap Q) = (\bigcup_{z \in B} C_z) \cap Q$, and $x \in \overline{P_x}$ implies $x \in \overline{(\bigcup_{z \in B} C_z) \cap Q}$.

Part B: If Q is an open set then $\mu(C_x \cap Q) = \mu(C_x) - \mu(C_x \setminus Q)$; $\mu(C_{\sqcup B} \cap Q) = \mu(C_{\sqcup B}) - \mu(C_{\sqcup B} \setminus Q)$. The non-trivial part is to show $\mu(C_{\sqcup B} \cap Q) \leq \sup_{x \in B} \mu(C_x \cap Q)$. This follows from $\mu(C_{\sqcup B}) = \sup_{x \in B} \mu(C_x)$ by Part A and $\mu(C_{\sqcup B} \setminus Q) \geq \sup_{x \in B} \mu(C_x \setminus Q)$ by monotonicity of μ.

□

Lemma 6.2 *Assume that μ is a continuous valuation and B is a directed subset of A. Then $\mu(I_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(I_x \cap Q))$, were Q is an open or closed subset of A.*

Remark: Co-continuity is not needed here.

Proof. During the proof of Lemma 12 in [2] it was established that $I_{\sqcup B} = \bigcup_{x \in B} I_x$ (think about $k \in A$ and observe that that proof goes through).

Part A: If Q is an open set $I_{\sqcup B} \cap Q = (\bigcup_{x \in B} I_x) \cap Q = \bigcup_{x \in B}(I_x \cap Q)$ and $\mu(I_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(I_x \cap Q))$ is a direct consequence of continuity of μ.

Part B: If Q is closed let us think about $\mu(I \cap Q)$ as $\mu(I) - \mu(I \setminus Q)$, where both I and $I \setminus Q$ are open sets. The rest is similar to Part B of the proof of the previous Lemma.

□

Lemma 6.3 *Assume that μ is a strongly non-degenerate valuation. Then the $\mu Info$-structure axiom* **Scott open sets are relaxed metric open** *holds.*

Remark: Neither continuity, nor co-continuity required, and even strong non-degeneracy condition can be probably be made weaker (see the next Section).

Proof. Consider a Scott open set U and $x \in U$. We would like to find $\delta > 0$, such that $\mu(C_x) - \mu(C_x \cap C_y) < \delta \Rightarrow y \in U$ (and hence, $B_{x,\delta+u(x,x)} \subseteq U$). Using $x = \sqcup P_x$, where $P_x = \{y \mid y \prec\prec x\}$ is a directed set, obtain $\exists y \prec\prec x.\ y \in U$ (observe that all this works for x — compact, $y = x$). Consider a Scott open set $\{z \in A \mid y \prec\prec z\} \ni x$ and a Scott open set $A \setminus C_x$. Then $x \notin (A \setminus C_x)$, hence $(A \setminus C_x) \cup \{z \in A \mid y \prec\prec z\} \supset (A \setminus C_x)$ and by strong non-degeneracy $\mu((A \setminus C_x) \cup \{z \in A \mid y \prec\prec z\}) - \mu(A \setminus C_x) = \epsilon > 0$. We take $\delta = \epsilon$ and prove that $\mu(C_x) - \mu(C_x \cap C_z) < \delta$ implies $y \sqsubseteq z$, and thus $z \in U$. Indeed, assume $y \not\sqsubseteq z$. Then we have that $y \notin C_z$, thus $\{z \mid y \prec\prec z\} \subseteq (A \setminus C_z)$, thus $\mu((A \setminus C_x) \cup (A \setminus C_z)) \geq \mu(A \setminus C_x) + \epsilon$, implying $\mu(C_x) - \mu(C_x \cap C_z) \geq \delta$.

□

7 Examples and Non-degeneracy Issues

In this section we show some examples of "nice" partial metrics, based on valuations for vertical and interval domains of real numbers. Some of these valuations are strongly non-degenerate, while others are not, yet all examples are quite natural. This prompts us to look for weaker non-degeneracy properties, which still would preserve Lemma 6.3, in our future research.

Consider the example from Subsection 3.2. The partial metric, based on the strongly non-degenerate valuation μ' of that example would be $u'(x,y) = (1-\min(x,y))/(1+\epsilon)$, if $x, y > 0$, and $u'(x,y) = 1$, if x or y equals to 0. However, another nice valuation, μ'', can be defined on the basis of μ of Subsection 3.2: $\mu''((x,1]) = \mu((x,1]) = 1-x$, $\mu''([0,1]) = 1$. μ'' is not strongly non-degenerate, however it yields nice partial metric $u''(x,y) = 1 - \min(x,y)$, yielding the Scott topology.

Now we consider several valuations and distances on the domain of interval numbers located within the segment $[0,1]$. This domain can be thought of as a triangle of pairs $< x, y >$, $0 \leq x \leq y \leq 1$. Various valuations can either be concentrated on $0 < x \leq y < 1$, or on $x = 0$, $0 \leq y \leq 1$ and $y = 1$, $0 \leq x \leq 1$, or, to insure non-degeneracy, on the both of these areas with an extra weight at $< 0, 1 >$.

Among all these measures, the classical partial metric $u([x,y],[x',y']) = \max(y,y') - \min(x,x')$ results from the valuation accumulated at $x = 0$, $0 \leq y \leq 1$, and $y = 0$, $0 \leq x \leq 1$, namely $\mu(U) = (Length(\{x = 0, 0 \leq y \leq$

$1\} \cap U) + Length(\{y = 1, 0 \leq x \leq 1\} \cap U))/2$. Partial metrics denerated by strongly non-degenerate valuations typically contain quadratic expressions.

It is our current feeling, that instead of trying to formalize weaker non-degeneracy conditions, it is fruitful to build a $\mu Info$-structure based on $\mathcal{I} = [0,1] + [0,1]$ in situations like this.

8 Conclusion

Summarizing, we obtained the following results. We introduced notions of co-continuous valuations and CC-valuations, and built CC-valuations for all continuous dcpo's with countable bases. Given such valuation, we built a new construction of partial and relaxed metrics for all continuous Scott domains, improving a construction known before.

Let us focus on the open problems now. Some of the open problems, which in some form where in our focus before the results presented in this paper can be found in Sections 5.1 and 7.

The key open problem is to learn to construct not just topologically correct, but canonical measures and relaxed metrics for higher-order functional domains and reflexive domains, and also to learn how to compute these measures and metrics quickly.

The other newly emerged open problems follow.

1) Is it possible to present every continuous partial metric yielding the Scott topology on a continuous Scott domain (may be, with countable basis) via a CC-valuation on A as $u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y))$, where $Info(x) = C_x$, $Neginfo(x) = I_x$ or using some other reasonable formulas for $Info(x)$ and $Neginfo(x)$. If not, what are the necessary and /or sufficient conditions for this. This can be also viewed as the question about our ability to define measures via partial metrics.

2) Are there any other reasonable formulas besides $Info(x) = C_x$, $Neginfo(x) = I_x$, which would also yield a good result?

3) Is it possible to extend the results of the second part of our paper beyond bounded complete domains? The obstacles to such extension seem to be related only to our use of I_x as $Neginfo(x)$. It is easy to come up with examples when I_x is not an open set, and $I_{\sqcup B} \neq \cup_{x \in B} I_x$ for a directed set B, unless bounded completeness is imposed.

4) Is it possible to treat relaxed metrics where upper bounds do not satisfy axioms of partial metrics via valuations?

5) We present CC-valuations as upper bounds of non-CC-valuations. Wee also know that it is possible to present any continuous valuations (including CC-valuations) via upper bounds of simple valuations. Can this construction be translated directly to partial metrics, and what would be the result?

Aknowledgements

The authors benefited from discussions with Michael Alekhnovich, Klaus Keimel, Harry Mairson, Simon O'Neill, and Joshua Scott. They thank Gordon Plotkin for helpful references. They are especially thankful to Abbas Edalat for his suggestion to think about continuous valuations instead of measures in this context, and to Alexander Artemyev for his help in organizing this joint research effort.

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