Partial Metrics and Quantale-valued Sets (Extended Abstract)

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Abstract

It is observed that the axioms for partial metrics with values in quantales coincide with the axioms for Q-sets (M-valued sets, sets with fuzzy equality, quantale-valued sets) for commutative quantales. Ω-sets correspond to the case of partial ultrametrics.

Keywords: sheaves, fuzzy sets, partial metrics, Scott domains, quantales, intuitionistic logic, generalized distances, generalized equality

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1 Partial metrics

The generalized distances without the axiom $p(x,x) = 0$ are frequently used to define non-Hausdorff topologies. They have various applications in semantics of programming languages and in the theory of computations with real numbers.

Kopperman, Matthews, and Pajoohesh generalize partial metrics so that they would take their values in quantales rather than in non-negative reals as follows.

The quantale V in question is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a+b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a+c \geq b\}$. Additional conditions are imposed.

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

2 Quantale-valued sets/fuzzy equality

Quantale-valued sets arose from the need to give solid foundation to the theory of fuzzy sets. They generalize Ω-sets introduced by Fourman and Scott in connection with sheaves and logic. Ulrich Höhle defines an M-valued set as follows.

The quantale M in question is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \leq b\}$. Additional conditions are imposed.

An M-valued set is a set X equipped with a fuzzy equality, which is a map $E : X \times X \to M$ subject to the axioms

- $E(x,y) \leq E(x,x) \wedge E(x,y)$
- $E(x,y) = E(y,x)$
- $E(x,y) + (E(y,y) \Rightarrow E(y,z)) \leq E(x,z)$

It's easy to see that the only difference between an M-valued set and a set with a V-pseudopmetric, besides the particular restrictions imposed on the quantale, is in notation: multiplicative vs. additive, the adjoint operation is denoted differently and the order of its arguments is switched, and the quantale order is reversed.

Hence the notions of an M-valued set and a set with a V-pseudopmetric coincide.

In [1] we provide further details and references related to the present work including the correspondence between Ω-sets and partial ultrametrics.

References

[1] Bukatin, M., R. Kopperman, S. Matthews, and H. Pajoohesh, "Partial Metrics and Quantale-valued Sets", Preprint, URL: http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html.