Michael Bukatin presents:

Partial Metrics and Quantale-valued Sets

by Michael Bukatin, Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh

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1. Introduction

"(some) generalized distances = (some) generalized equalities"

The axioms for partial metrics with values in quantales coincide with the axioms for Q-sets (M-valued sets, sets with fuzzy equality, quantale-valued sets) for commutative quantales.

Ω-sets correspond to the case of partial ultrametrics.

Partial metrics usually occur in context of domains for denotational semantics, and quantale-valued sets usually occur in context of sheaves.

Domains for denotational semantics and sheaves are two different (but related in spirit) approaches to the theory of partially defined elements.

Both approaches have applications to computability in analysis. E.g. domains were used by Edalat to handle Riemann integrals of functions defined on fractals, and then by many scientists to talk about computability of various structures in analysis.

The authors of the present paper come from the domain side. Therefore, speaking about applications of sheaves to computability in analysis I should mention, for example, the connections between realizability toposes and recursive analysis without claiming to understand them at this point.

"Common origins" (in some sense).

The seminal paper by Dana Scott, "Continuous Lattices", which many people consider to be the "official" start of domain theory was published in a volume of Springer Lecture Notes in Mathematics essentially dedicated to sheaves:

"Toposes, Algebraic Geometry and Logic", LNM, v.274, 1972

The approach to sheaves based on Ω-sets also originated with Dana Scott (and Michael Fourman, and also Denis Higgs).

Nevertheless, domains and sheaves are usually treated as technically unrelated for a number of reasons. Not many articles and books mention them together.

A notable exeption here is synthetic domain theory. I am not sufficiently familiar with this theory yet to elaborate.

The observation I am presenting today might enable further interesting interactions between domains and sheaves.

To summarize the introduction

The theory of generalized distances without the axiom $p(x,x) = 0$, known as **partial metrics**, is closely related to the "domains approach".

The theory of generalized equalities, namely non-reflexive equalities with values in complete Heyting algebras and quantales, is closely related to the "sheaves approach".

We observe that the axioms for partial metrics with values in quantales coincide with the axioms for quantale-valued sets for commutative quantales.

In other words, generalized symmetric non-reflexive distances and generalized equalities (quantale-valued fuzzy equalities) coincide.

2. Partial metrics

The generalized distances without the axiom $p(x, x) = 0$ in the context of analyzing deadlock in lazy data flow computations were studied by Steve Matthews in his Ph.D. thesis. Then certain axioms regaining some of what is lost by dropping $p(x, x) = 0$ were added, namely, small self-distances, $p(x, x) \leq p(x, y)$, and the sharpened form of triangular equality, $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$, introduced by Steve Vickers.

Michael Bukatin and Joshua Scott studied generalized distances on Scott domains and noted that axiom $p(x,x) = 0$ is incompatible with Scott continuity (or computability) of distances in question. It should be noted that in the similar fashion the axiom $x = x$, which can be rewritten as $Eq(x,x) = true$, prevents the traditional equality from being Scott continuous (or computable).

Because Bukatin and Scott were interested in Scott continuity of the resulting distances and, hence, needed monotonicity, their domain of numbers was the set of non-negative reals (with added $+\infty$) with the reverse order: 0 was the largest element, and $+\infty$ was the smallest.

However, the traditional view of 0 as the smallest possible distance remained more prevalent in the partial metrics research community, and when Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh generalized partial metrics so that they would take their values in quantales rather than in non-negative reals, the axioms looked as follows.

The quantale V in question was a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element was the bottom element 0. The right adjoint to the map $b \mapsto a + b$ was defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions were imposed.

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ were

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

3. Quantale-valued sets

Ω-sets were introduced by Michael Fourman and Dana Scott (cf. also Denis Higgs). If Ω is a complete Heyting algebra, an Ω-set A is a set equipped with an Ω-valued generalized equality, $E : A \times A \to \Omega$, subject to axioms $E(a,b) = E(b,a)$ and $E(a,b) \wedge E(b,c) \leq E(a,c)$.

Fourman and Scott also introduced a mechanism of *singletons*, which was used to define the notion of *complete Ω-set* and to establish that complete Ω-sets and sheaves over complete Heyting algebra Ω are essentially the same thing.

[see Fourman M. P., D. S. Scott, *Sheaves and Logic*, in "Applications of Sheaf Theory to Algebra, Analysis, and Topology," Lecture Notes in Mathematics, **753**, Springer, 1979, pp. 302–401.]

Returning to partial metrics, from the symmetry and the ultrametric triangle inequality, $p(x,z) \leq max(p(x,y), p(y,z))$, one can obtain $p(x,x) \leq max(p(x,y), p(y,x)) = p(x,y)$.

$p(x,z) \leq max(p(x,y), p(y,z))$ means $p(x,z) \leq p(x,y)$ or $p(x,z) \leq p(y,z)$. Consider $p(x,z) \leq p(x,y)$. We know that $p(y,y) \leq p(y,z)$, so $p(x,z)+p(y,y) \leq p(x,y)+p(y,z)$, and we obtain the Vickers form of triangle inequality. Consider $p(x,z) \leq p(y,z)$. We know that $p(y,y) \leq p(x,y)$, and we again obtain $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

So utrametrics without axiom $p(x,x) = 0$ obey both extra axioms of partial metrics. This justifies the term *partial ultrametrics* and also tells us that we should consider Ω-equality as partial ultrametric with more general values than non-negative reals.

This motivated our search in the literature for generalizations of Ω-equality beyond complete Heyting algebras.

[Wagner, K. R., "Solving Recursive Domain Equations with Enriched Categories," Ph.D. thesis, School of Computer Science, Carnegie Mellon University, Pittsburg, 1994.]

[Wyler O., "Lecture Notes on Topoi and Quasitopoi," World Scientific Publishing Company, 1991.]

[Höhle U., *M-valued sets and sheaves over integral, commutative* **cl**-*monoids*, in S.E. Rodabaugh et al, editors, "Applications of Category Theory to Fuzzy Subsets," Kluwer Academic Publishers, Dordrecht, Boston, London, 1992, pp. 33–72.]

Ulrich Höhle was motivated by the need to give solid foundation to fuzzy set theory (and, in particular, to the uses of such logical systems as Lukasiewicz logic). His definition of an M-valued set looked as follows.

The quantale V in question was a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element was the top element 1. The right adjoint to the map $b \mapsto a * b$ was defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a*c \leq b\}$. Certain additional conditions were imposed.

An M-valued set was a set X equipped with a map $E : X \times X \to M$ subject to the axioms

- $E(x,y) \leq E(x,x) \cancel{\wedge E(x,y)}$

- $E(x,y) = E(y,x)$

- $E(x,y) + (E(y,y) \Rightarrow E(y,z)) \leq E(x,z)$

It is easy to see that the only difference between an M-valued set and a set with a V-pseudopmetric, besides the particular restrictions imposed on the quantale, is in notation: multiplicative vs. additive, the adjoint operation is denoted differenty and the order of its arguments is switched, and the quantale order is reversed.

So the notions of an M-valued set and a set with a V-pseudopmetric coincide.

This is the main observation we would like to communicate today.

Further observations

An M-valued set is called separated iff the axiom

$$E(x,x) \vee E(y,y) \leq E(x,y) \text{ implies } x = y$$

holds. It's easy to see that this is equivalent to the separation axiom, making partial pseudometric p into a partial metric (V-pmetric):

$$\forall x, y \in X.\, x = y \text{ iff } p(x,y) = p(x,x) = p(y,y).$$

There are further parallels, e.g. similar extra conditions on quantales are imposed and studied, such as existence of *halves* in V vs. existence of *square roots* in M, etc, etc.

We should also say that Höhle provides a generalization of singletons (Fourman, Scott) and and develops a comprehensive theory relating M-sets to a version of sheaves.

For the current state of the quantale-valued sets, including generalizations to non-commutative quantales, see

Höhle U., "Sheaves and quantales," Preprint

`http://www.math.uni-wuppertal.de/~hoehle/publications/preprints.html`

and references therein.

4. Some applications

I'll mention two small applications today, one on the sheaf side, and one on the domain side. Starting with the sheaf side:

If Ω is a complete Heyting algebra, an Ω-set A is a set equipped with an Ω-valued generalized equality, $E : A \times A \to \Omega$, subject to axioms $E(a,b) = E(b,a)$ and $E(a,b) \wedge E(b,c) \leq E(a,c)$.

Consider

$$F(a,b) = (E(a,a) \Rightarrow E(a,b)) \wedge (E(b,b) \Rightarrow E(a,b)).$$

This is also an Ω-equality, and moreover (A,F) is a global Ω-set: $F(a,a)$ is the top element of Ω for all a. Basically, F is not just a quantale-valued partial ultrametric, but a quantale-valued ultrametric.

Given a presheaf A of sets over a complete Heyting algebra Ω, one can define an Ω-equality

$$\cancel{E(a,b) = \bigvee\{p \in \Omega \mid a \upharpoonright p = b \upharpoonright p\}}$$

$$E(a,b) = \bigvee\{p \leq E(a) \wedge E(b) \mid a \upharpoonright p = b \upharpoonright p\}$$

and then define a quantale-valued ultrametric F as above.

Consider $p \leq q \in \Omega$. Then the restriction map from $\{a \in A \mid E(a,a) = q\}$ to $\{a \in A \mid E(a,a) = p\}$ is a non-expansive map with respect to F.

So A becomes a presheaf of quantale-valued ultrametric spaces with non-expansive maps as morphisms.

An "application" on the domain side:

The question often arises: what is the right notion of morphism in a given context. E.g. for metric spaces, should one consider all continuous maps, or just non-expansive maps, etc.

The sheaf analogy suggests that the category of partial metrics with values in a given quantale and **weight-preserving** non-expansive maps is an interesting category to consider.