Some examples of partial metrics
(quantale-valued sets)

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This is an appendix to "Partial Metrics and Quantale-valued Sets" preprint.

1 Reminder

There is an order duality between thinking in terms of generalized distances (0 is the smallest number) and generalized equalities ("true", also known as 0 in the additive notation, also known as 1 in the multiplicative notation, is the largest number).

So something is decreasing from the metric point of view if and only if it is increasing from the logical point of view (generalized equalities point of view).

2 A partial ultrametric on sequences

The quantale in question is $[0, +\infty]$, with the binary operation **max**.

The space consists of finite and countable sequences of letters of some alphabet. The distance between two sequences is 2^{-n}, where n is the length of their common prefix.

The distance between a finite sequence of length n and itself is 2^{-n}. The distance between an infinite sequence and itself is 0.

It is easy to check that this is a partial ultrametric (an Ω-set).

3 A partial metric on interval numbers

The quantale in question is $[0, +\infty]$, with the binary operation $+$.

The space consists of pairs of real numbers, $a \leq b$. We think about these pairs as representing segments $[a, b]$. The distance between two segments, $[a, b]$ and $[c, d]$, is $max(b, d) - min(a, c)$.

The distances between a segment $[a, b]$ and itself is $b - a$, this distance is zero if and only if $a = b$.

This distance is not a partial ultrametric, but it is a partial metric.

In particular, let's verify $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Take $x = [a, b], y = [c, d], z = [e, f]$.

Then $p(x, z) + p(y, y) = max(b, f) - min(a, e) + d - c$. Meanwhile $p(x, y) + p(y, z) = max(b, d) - min(a, c) + max(d, f) - mix(c, e)$

We want to show that $max(b, f) + d \leq max(b, d) + max(d, f)$. Without the loss of generality, assume $b \leq f$. We want to show $f+d \leq max(b, d)+max(d, f)$. Case 1: $d \leq b$: indeed $f + d \leq b + f$. Case 2: $b \leq d \leq f$: indeed $f + d \leq d + f$. Case 3: $f \leq d$: indeed $f + d \leq d + d$.

By the symmetry $-min(a, e) - c \leq -min(a, c) - mix(c, e)$.

4 An Ω-set of partially defined functions

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Consider a topological space X. The quantale in question is the topology, that is the set of all open subsets U of X. The role of "truth", or 0, plays the open set X itself. The binary operation is the intersection, $\cap$.

Select some codomain Y. The space in question consists of some class of functions $f : U \to Y$, where U is an open subset of X. The distance between two functions f and g is the interior of $\{x \in X \mid f(x) = g(x)\}$.

The distance between a function and itself is its own domain of definition.

So the distance between a function and itself is "zero" (the whole space X) if and only if this function is complete defined.

To verify that this is an Ω-set (a partial ultrametric), we need to show that $p(f, g) \cap p(g, h) \subseteq p(f, h)$ (note the order duality between the metric and logical points of views). The check is straightforward: if x belongs to an open set U, on which $f = g$, and x belongs to an open set V, on which $g = h$, then x also belongs to an open set $U \cap V$, on which $f = h$.