Relaxed Metrics, Maximal Points, and Negative Information*

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September 24, 1998

Abstract. In our earlier paper, we presented a method of obtaining partial and relaxed metrics from co-continuous and bi-continuous valuations for any continuous Scott domain A via formalism of μ*Info*-structures. Certain pathologies in the behavior of $I_x = \{y \in A \mid \{x, y\}$ is unbounded$\}$ precluded us from extending this method beyond bounded complete domains.

In this paper, we obtain meaningful Scott continuous relaxed metrics for continuous dcpo's by replacing I_x with $Neginfo(x) = \mathrm{Int}(J_x)$, where $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$. This result can be understood in terms of interplay between *negation duality* and *Stone duality.*

This technique works especially well, when a continuous dcpo A satisfies the *Lawson condition* — the relative Scott and Lawson topologies on $Total(A)$ are equal. We obtain that the Lawson condition is equivalent to the property $\forall x \in Total(A).\, I_x = J_x$. This implies that when the Lawson condition holds, an induced metric on $Total(A)$ results.

1 Introduction

Partial orders equipped with Scott topology remain the most popular continuous model of computation (see [16, 8] for the compact presentation of many relevant issues). The last decade was marked by rapid development of analysis over such domains. One should especially mention the theory of generalized metrics over domains (see [11, 13, 2] and references therein), the theory of valuations and measures over domains (see [4, 10, 3] and references therein), and various deep connections between generalized metrics and valuations over domains [14, 3, 15].

The work on Scott continuous relaxed metrics [2, 3], co-continuous and bi-continuous valuations [3, 10], and their interplay [3] increased hopes for eventual applicability of functional analysis methods to non-numerical software engineering.

In somewhat dual development, the nineties were marked by the emergence of domain-based computational models for various structures from classical mathematics [4]. Generalized metrics and valuations over domains played the central role in this development as well. The remarkable Lawson condition [12], that relative Scott and Lawson topologies on the space of total elements coincide, is frequently imposed on domain-based models of topological spaces [5].

Also certain connections between domain theory and the theory of tolerance spaces started to emerge recently [17, 19].

Somewhat surprisingly, relaxed metrics, bi-continuous valuations, Lawson condition, and tolerances come into elegant interplay in the present paper.

*An MFPS Proceedings Submission — version 0.6

1.1 Our previous work

In [3], we present a method of obtaining partial and relaxed metrics from co-continuous and bi-continuous valuations on the systems of Scott open sets of domains via the mechanism of μ*Info*-structures. Specifically, we take a normalized valuation μ on the system of Scott open sets of a domain A, extend it to the ring of sets via finite unions, intersections, and compliments, and associate measurable sets $Info(x) \subseteq A$ and $Neginfo(x) \subseteq A$ with every $x \in A$. Then we define the interval-valued function $\rho(x,y) = [l(x,y), u(x,y)]$, where $l(x,y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x))$, and $u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y))$. If the axioms of *deficient* μ*Info*-structure hold (see Section 2.4), we obtain that ρ satisfies axioms of relaxed metric, u satisfies axioms of partial metrics, relaxed metric ρ is a Scott continuous function from the domain A to the domain of interval numbers (*continuity property*), and the resulting relaxed metric topology coincides with Scott topology.

Typically, we also study the set of maximal elements, $Total(A)$, of our domain and would like the *totality property*, $\forall x \in Total(A).\, Info(x) \cup Neginfo(x) = A$, to hold. This totality property would imply, that for $x, y \in Total(A)$, the upper and lower bounds of the resulting relaxed metric coincide (that is, $u(x,y) = l(x,y)$). This would make $Total(A)$ a metric space, and the resulting metric topology would coincide with relative Scott topology on $Total(A)$.

In particular, given a co-continuous normalized valuation μ on a continuous dcpo A, we can build Scott continuous relaxed metric $[0, 1-\mu(C_x \cap C_y)]$. Here $Info(x) = C_x = \{y \in A \mid y \sqsubseteq x\}$, $Neginfo(x) = \emptyset$, and issues of totality are ignored.

If, in addition, valuation μ is continuous, and domain A is bounded complete, we can build a more sophisticated Scott continuous relaxed metric, $[\mu(C_x \cap I_y) + \mu(C_y \cap I_x), 1 - \mu(C_x \cap C_y) - \mu(I_x \cap I_y)]$, by taking $Neginfo(x) = I_x = \{y \in A \mid \{x,y\} \text{ is unbounded}\}$. In both cases, here and everywhere in this paper, the upper bounds of the relaxed metrics are partial metrics introduced by Matthews [13]. When valuation μ is also strongly non-degenerate, the resulting relaxed metric topology coincides with Scott topology on A. Moreover, the totality property, $\forall x \in Total(A).\, C_x \cup I_x = A$, holds with all its ramifications. It is known how to build co-continuous, normalized, continuous valuations for all continuous dcpo's A, and how to make these valuations strongly non-degenerate, when A has a countable basis [10, 3].

1.2 Negation Duality and Problems with I_x

Certain pathologies in the behavior of I_x do not allow us to use the formula above when A is not bounded complete. It is convenient to use *negation duality*, $x \in I_y \Leftrightarrow y \in I_x$, when analyzing the behavior of I_x. Specifically, one can consider directed sets $B \subseteq A$, and by considering $x = \sqcup B$ and $x = b \in B$, obtain the following Lemma:

Lemma 1.1 *For any dcpo A, ($\forall y \in A.\, I_y$ is Scott open) $\Leftrightarrow$ ($\forall B \subseteq A.\, B$ is directed $\Rightarrow I_{\sqcup B} = \bigcup_{b \in B} I_b$) .*

One can informally restate this, by saying that all I_y are (Scott) observable /citeSmyth, if and only if I is a Scott continuous function $A \to \mathcal{P}(A)$.

We will see that for some continuous dcpo's certain I_y's are not Scott open. This would be possible to tolerate, since, as we will see later, co-continuity of valuations allows to extend them to Alexandrov open sets, and any I_y is still Alexandrov open. However, the corresponding breakdown of equality $I_{\sqcup B} = \bigcup_{b \in B} I_b$ for directed sets B cannot be tolerated, because it tend to lead to the loss of Scott continuity for the resulting relaxed metrics.

We could have replaced I_x with $\mathrm{Int}(I_x)$ (here and everywhere in this paper, interiors are being taken in Scott topology), to rectify the problem of I_x being not Scott open, but this does nothing to fix the broken continuity property. Somewhat surprisingly, the *negation duality* helps us to resolve this problem.

1.3 Negation Duality and Stone Duality

We consider equality $I_x = \{y \in A \,|\, x \in I_y\}$, which is just another way to state the negation duality. Then, instead of taking $\mathrm{Int}(I_x)$, we consider $J_x = \{y \in A \,|\, x \in \mathrm{Int}(I_y)\}$. Then the continuity property for the resulting relaxed metric will be restored. It is going to be technically convenient to replace I_x not with J_x, but with $\mathrm{Int}(J_x)$, but we do not think that this feature is principal.

We will see that the reasons for J_x to work in this situation can be best understood in terms of Stone duality. In particular, the map $y \mapsto \mathrm{Int}(I_y)$ gives rise to a map of the generators $\uparrow\{y\}$ of the frame of Scott open sets of the powerset of A (ordered by inclusion, the original ordering on A is ignored) to the frame of open sets of the domain A. Function $x \mapsto J_x$ turns out to be a Scott continuous function $A \to \mathcal{P}(A)$, which is dual to the function $\mathcal{O}(\mathcal{P}(A)) \to \mathcal{O}(A)$ obtained by extending map $\uparrow\{y\} \mapsto \mathrm{Int}(I_y)$ onto the frame $\mathcal{O}(\mathcal{P}(A))$. We should emphasize here, that what is going on in this paper is a rather subtle interplay of two *different* dualities — negation duality and Stone duality — none of which seems to be reducible to another.

We will also see, that the function $x \mapsto J_x$ is the largest "negative" Scott continuous function $A \to \mathcal{P}(A)$, where the powerset of A, $\mathcal{P}(A)$, is ordered by the set-theoretic inclusion. This means that if $f : A \to \mathcal{P}(A)$ is another Scott continuous function, such that for all $x \in A$, $f(x) \subseteq I_x$, then for all $x \in A$, $f(x) \subseteq J_x$.

1.4 Lawson Condition

In general we only get deficient μ*Info*-structures via the use of $Neginfo(x) = J_x$ or $Neginfo(x) = \mathrm{Int}(J_x)$. Namely, the totality property does not hold in general, and thus we still do not get an induced metric on $Total(A)$. However, in this situation it helps to impose the *Lawson condition*, that relative Scott and Lawson topologies on $Total(A)$ are equal.

The Lawson condition was introduced in [12], and is widely used lately. As Lawson writes in the Introduction to [12], "this turns out to be a very fruitful notion that permit great generality, but at the same time permits the derivation of many important structure results". This condition is now a standard part of the notion of a *computational model* for a topological space [5].

In this paper, we present two equivalent formulations of the Lawson condition: $\forall x \in A.\, I_x \cap Total(A) = \mathrm{Int}(I_x) \cap Total(A)$ and $\forall x \in Total(A).\, J_x = I_x$.

The second of these equivalent formulations implies that when the Lawson condition holds for the domain A, for any maximal element x, $\mathrm{Int}(J_x) = J_x = I_x$, the totality property, $C_x \cup Neginfo(x) = A$, holds, and the induced metric on $Total(A)$ results. Of course, the resulting metric topology is the same as relative Scott and Lawson topologies on $Total(A)$.

1.5 Polish Spaces

Lawson has shown in [12], that for every continuous dcpo A with countable basis, Lawson condition implies that $Total(A)$ is a Polish space, i.e. that it is homeomorphic to a complete, separable metric space.

However, since completeness of metric spaces is not a topological invariant, this does not mean that metrics obtained by our present methods have to be complete. Indeed, our present methods, based on assignment of converging systems of finite weights for the case of algebraic domains, yield a non-complete metric on $Total(E')$ for the domain E' of Section 6.

This raises a lot of open questions, ranging from the question of when our construction yields a complete metric space to the question of whether methods of Edalat and Heckmann, used to approximate complete metric spaces (see [7] for the variant of their approach using partial metrics and, thus, closest to our methods), can be extended to describe certain non-complete metric spaces, like an open interval of the real line or sets like $\{1/2, 1/4, 1/8, \ldots\}$.

1.6 Tolerance Spaces

Lately, deep connections between the material presented in the present paper and such notions as tolerance spaces and weakly Hausdorff spaces [17, 19] have started to emerge. While it seems proper to postpone the detailed exploration of these connections until another occasion, we feel the need to outline some of them here briefly.

First of all, Bob Flagg noted, that our results from Section 4 can be generalized to continuous dcpo's with compact Lawson topology. Mike Smyth observed that this is a corollary of the following two facts. The first fact is that the conditions that all I_x are Scott open, that $I = J$, and that the continuous dcpo is weakly Hausdorff are equivalent. The second fact is that continuous dcpo's with compact Lawson topology are weakly Hausdorff [17].

Since $I = J$ is equivalent to the space in question being weakly Hausdorff, and since the Lawson condition can be reformulated as $I_x = J_x$ for maximal x, we can offer an alternative name for the spaces, satisfying the Lawson condition — *very weakly Hausdorff spaces*.

Using this terminology, we can say that one of the discoveries of this paper is that continuous dcpo's do not have to be weakly Hausdorff to be satisfactorily relaxed metrizable by our methods, but it is enough to require that they be very weakly Hausdorff.

1.7 Structure of the Paper

In Section 2 we describe basic notions and results from [3] regarding co-continuous valuations, relaxed metrics, and μ*Info*-structures. In Section 3 we present the mechanism of obtaining relaxed metrics from valuations from [3], adjusting this presentation to our current needs.

In Section 4 we show that the old formulas, based on I_x, work for the class of *coherently continuous* dcpo's with countable basis, because I_x's are Scott open for this class of domains.

In Section 5 we analyze the pathologies of behavior of I_x on a specific example. We then study the properties of J_x, which serves as a replacement for I_x, and explain those properties from the viewpoint of Stone duality.

In Section 6 we find equivalent formulations of the Lawson condition and use these formulations to establish the totality property of the resulting μ*Info*-structures with its ramifications for the induced metrics on $Total(A)$.

In Conclusion we focus on some of the open issues.

2 Basic Notions

In this Section we recall basic notions and facts, which can be found in [3].

2.1 Continuous Scott Domains

Recall that a non-empty partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. A poset, $(A, \sqsubseteq)$, is a *dcpo* if it has the least element, $\perp$, and for any directed $S \subseteq A$, the least upper bound $\sqcup S$ of S exists in A. A set $U \subseteq A$ is *Scott open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$ and for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$. The Scott open subsets of a dcpo form the *Scott topology*.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. $f : A \to B$ is (Scott) continuous iff it is monotonic ($x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$) and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

We define continuous Scott domains in the spirit of [8]. Consider a dcpo $(A, \sqsubseteq)$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$. An element x, such that $x \ll x$, is called *compact*. We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B$ exists.

Consider a set $K \subseteq A$. Notice that $\perp_A \in K$. We say that a dcpo A is a *continuous dcpo* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements. A continuous, bounded complete dcpo is called a *continuous Scott domain*.

2.2 CC-valuations

Consider a topological space $(X, \mathcal{O})$, where $\mathcal{O}$ consists of all open subsets of X.

Definition 2.1. A function $\mu : \mathcal{O} \to [0, +\infty]$ is called *valuation* if

1. $\forall U, V \in \mathcal{O}.\ U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
2. $\forall U, V \in \mathcal{O}.\ \mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
3. $\mu(\emptyset) = 0$.

Definition 2.2. A valuation μ is *bounded* if $\mu(X) < +\infty$. A valuation μ is *normalized* if $\mu(X) = 1$.

Remark: If a valuation μ is bounded and $\mu(X) \neq 0$, then it is easy to replace it with a normalized valuation $\mu'(U) = \mu(U)/\mu(X)$.

Definition 2.3. Define a *directed system of open sets*, $\mathcal{U} = \{U_i,\ i \in I\}$, as satisfying the following condition: for any finite number of open sets $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_{i_1} \subseteq U_i, \cdots, U_{i_n} \subseteq U_i$.

Definition 2.4. Dually, define a *filtered system of open sets*, $\mathcal{U} = \{U_i,\ i \in I\}$, as satisfying the following condition: for any finite number of open sets $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ there is U_i, $i \in I$, such that $U_i \subseteq U_{i_1}, \cdots, U_i \subseteq U_{i_n}$.

Definition 2.5. A valuation μ is called *continuous*, when for any directed system of open sets $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i)$.

Definition 2.6. A valuation $\mu : \mathcal{O} \to [0, +\infty]$ is *strongly non-degenerate*, if $\forall U, V \in \mathcal{O}.\ U \subset V \Rightarrow \mu(U) < \mu(V)$.[1]

Definition 2.7. A valuation μ is called *co-continuous* if for any filtered system of open sets $\{U_i,\ i \in I\}$

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I} \mu(U_i).$$

Definition 2.8. A continuous, normalized, strong non-degenerate, co-continuous valuation μ is called a *CC-valuation.*

Keimel also suggested to call continuous, co-continuous valuations *bi-continuous* [10].

We will use the following basic results arising in the theory of co-continuous valuations.

Lemma 2.1 (Border Lemma) *Consider an Alexandrov open set $B \subseteq A$. Then its interior in the Scott topology,* $\mathrm{Int}(B) = \{y \in A \mid \exists x \in B.\ x \ll y\}$. *Correspondingly, the border of B in the Scott topology,* $B \setminus \mathrm{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\ x \ll y)\}$

Theorem 2.1 *For any continuous dcpo A with a countable basis, there is a CC-valuation μ on the system of its Scott open sets.*

2.3 Relaxed Metrics

The motivations behind the notion of relaxed metric, its computational meaning and its relationships with partial metrics [13] were explained in [2]. Here we recall the definitions and basic properties.

[1] We use $U \subset V$ as an equivalent of $U \subseteq V\ \&\ U \neq V$.

The distance domain of non-negative interval numbers consists of pairs $\langle a, b\rangle$ (also denoted as $[a, b]$) of non-negative reals ($+\infty$ included), such that $a \leq b$. We denote this domain as R^I. $[a, b] \sqsubseteq_{R^I} [c, d]$ iff $a \leq c$ and $d \leq b$.

We can also think about R^I as a subset of $R^+ \times R^-$, where $\sqsubseteq_{R^+} = \leq$, $\sqsubseteq_{R^-} = \geq$, and both R^+ and R^- consist of non-negative reals and $+\infty$. We call R^+ a *domain of lower bounds*, and R^- a *domain of upper bounds*. Thus a distance function $\rho : A \times A \to R^I$ can be thought of as a pair of distance functions $\langle l, u\rangle$, $l : A \times A \to R^+$, $u : A \times A \to R^-$.

Definition 2.9. A symmetric function $u : A \times A \to R^-$ is called a *relaxed metric* when it satisfies the triangle inequality. A symmetric function $\rho : A \times A \to R^I$ is called a *relaxed metric* when its upper part u is a relaxed metric.

An *open ball* with a center $x \in A$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x, y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those points y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

Definition 2.10. A subset U of A is *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > u(x, x)$ such that $B_{x,\epsilon} \subseteq U$.

It is easy to show that for a continuous relaxed metric on a dcpo all relaxed metric open sets are Scott open and form a topology.

2.4 μ*Info*-structures

Some of the earlier known constructions of partial metrics can be understood via the idea that *common information* between elements x and y brings negative contribution to $u(x, y)$ (see [2, Section 8]). This can be further formalized as follows. Assume that there is a set $\mathcal{I}$ representing information about elements of a dcpo A. We choose a ring, $\mathcal{M}(\mathcal{I})$, of admissible subsets of $\mathcal{I}$ and introduce a measure-like structure, μ, on $\mathcal{M}(\mathcal{I})$. We associate a set, $Info(x) \in \mathcal{M}(\mathcal{I})$, with every $x \in A$, and call $Info(x)$ a set of (positive) information about x. We also would like to consider negative information about x, $Neginfo(x) \in \mathcal{M}(\mathcal{I})$, — intuitively speaking, this is information which cannot become true about x, when x is arbitrarily increased.

Definition 2.11. Given a dcpo A, the tuple of $(A, \mathcal{I}, \mathcal{M}(\mathcal{I}), \mu, Info, Neginfo)$ is called a *μInfo-structure* on A, if $\mathcal{M}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$ is a ring of subsets closed with respect to $\cap, \cup, \setminus$ and including $\emptyset$ and $\mathcal{I}$, $\mu : \mathcal{M}(\mathcal{I}) \to [0, 1]$, $Info : A \to \mathcal{M}(\mathcal{I})$, and $Neginfo : A \to \mathcal{M}(\mathcal{I})$, and the following axioms are satisfied:

1. **(VALUATION AXIOMS)**
   (a) $\mu(\mathcal{I}) = 1$, $\mu(\emptyset) = 0$;
   (b) $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
   (c) $\mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
2. **(*Info* AXIOMS)**
   (a) $x \sqsubseteq y \Leftrightarrow Info(x) \subseteq Info(y)$;
   (b) $x \sqsubset y \Rightarrow Info(x) \subset Info(y)$;
3. **(*Neginfo* AXIOMS)**
   (a) $Info(x) \cap Neginfo(x) = \emptyset$;
   (b) $x \sqsubseteq y \Rightarrow Neginfo(x) \subseteq Neginfo(y)$;

4. **(STRONG RESPECT FOR TOTALITY)**
   $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$;
5. **(CONTINUITY OF INDUCED RELAXED METRIC)**
   if B is a directed subset of A and $y \in A$, then
   (a) $\mu(Info(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Info(x) \cap Info(y))$,
   (b) $\mu(Info(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Info(x) \cap Neginfo(y))$,
   (c) $\mu(Neginfo(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Info(y))$,
   (d) $\mu(Neginfo(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Neginfo(y))$;
6. **(SCOTT OPEN SETS ARE RELAXED METRIC OPEN)**
   for any (basic) Scott open set $U \subseteq A$ and $x \in U$, there is an $\epsilon > 0$, such that $\forall y \in A.\ \mu(Info(x)) - \mu(Info(x) \cap Info(y)) < \epsilon \Rightarrow y \in U$.

We will also consider *deficient* $\mu Info$-structures, when the **strong respect for totality** axiom is not imposed.

2.5 Relaxed Metrics via $\mu Info$-structures

Consider a (possibly deficient) $\mu Info$-structure over domain A. Define the **induced relaxed metric**, $\rho(x,y) = [l(x,y), u(x,y)]$, via formulas:

$$u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y));$$

$$l(x,y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x)).$$

Now the following theorem holds even for deficient $\mu Info$-structures.

Theorem 2.2 *Function u is a partial metric. Function ρ is a continuous relaxed metric. The relaxed metric topology coincides with Scott topology.*

In the presence of the **strong respect for totality** axiom, the following theorem also holds.

Theorem 2.3 *For all x and y from $Total(A)$, $l(x,y) = u(x,y)$. Consider $d : Total(A) \times Total(A) \to \mathbf{R}$, $d(x,y) = l(x,y) = u(x,y)$. Then $(Total(A), d)$ is a metric space, and its metric topology is the subspace topology induced by Scott topology on A.*

3 Relaxed Metrics via CC-valuations

Let us recall a construction from [3]. This construction was done there for the case $N_x = I_x$ when A was bounded complete. However, bounded completeness was only used in order to establish the fact, that for all x, I_x is open, and that for any directed set B, $I_{\sqcup B} = \bigcup_{x \in B} I_x$.

Assume that there is a CC-valuation $\mu(U)$ on Scott open sets of a continuous dcpo A. Then it uniquely extends to an additive measure μ on the ring of sets generated by the system of Scott open sets. Define $\mathcal{I} = A$, $Info(x) = C_x$, $Neginfo(x) = N_x$, where N_x varies. It is easy to see that the valuation and *Info* axioms of $\mu Info$-structure hold.

Let us require that the *Neginfo* axioms of $\mu Info$-structure hold, that for all $x \in A$, N_x is Scott open, that for any directed B, $N_{\sqcup B} = \bigcup_{x \in B} N_x$, and that $x \in Total(A) \Rightarrow C_x \cup N_x = A$.

Thus we only need to establish $\mu Info$-structure axioms of **continuity of induced relaxed metrics** and **Scott open sets are relaxed metric open** in order to prove theorems 2.2

and 2.3 for our induced relaxed metric ($u(x,y) = 1 - \mu(C_x \cap C_y) - \mu(N_x \cap N_y)$, $l(x,y) = \mu(C_x \cap N_y) + \mu(C_y \cap N_x)$). These axioms are established by the Lemmas below.

You will also see that for such bare-bones partial metrics, as $u(x,y) = 1 - \mu(C_x \cap C_y)$, which are nevertheless quite sufficient for topological purposes and for domains with $\top$, only *co-continuity* matters, continuity is not important.

Lemma 3.1 *Assume that μ is a co-continuous valuation and B is a directed subset of A. Then $\mu(C_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(C_x \cap Q))$, where Q is a closed or open subset of A.*

Remark: Note that continuity of μ is not required here.

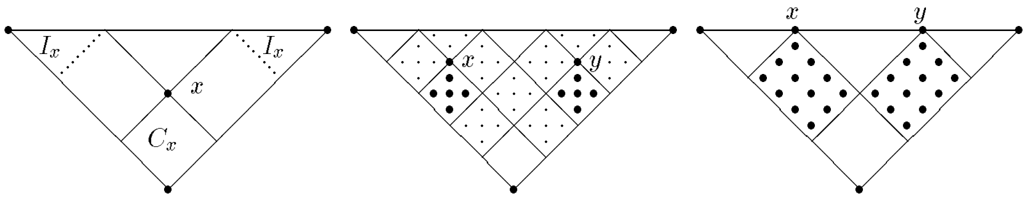
Lemma 3.2 *Assume that μ is a continuous valuation and B is a directed subset of A. Then $\mu(N_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(N_x \cap Q))$, where Q is an open or closed subset of A.*

Remark: Co-continuity is not needed here.

Lemma 3.3 *Assume that μ is a strongly non-degenerate valuation. Then the μInfo-structure axiom* **Scott open sets are relaxed metric open** *holds.*

Remark: Neither continuity, nor co-continuity required, and even the strong non-degeneracy condition can probably be made weaker.

It is helpful to visualize this situation via the following pictures, when $N_x = I_x$. These pictures show sets C_x and I_x, show the "area" accounting for $l(x,y)$ as shaded with large dots, the "area" accounting for $u(x,y)$ as shaded with all dots (large and small), and show why $l(x,y) = u(x,y)$, when $x, y \in Total(A)$.



4 When I_x Behaves Well

In this section we study cases, when for all $x \in A$, I_x is Scott open, or, equivalently, when for all directed $B \subseteq A$, $I_{\sqcup B} = \bigcup_{x \in B} I_x$. We already know, that this situation takes place for continuous Scott domains.

Another class of domains, for which these properties can be established, is the class of coherently continuous dcpo's with countable basis. The term "coherence" here is understood in the weak sense of [1] (weaker, than bounded completeness), and not in the strong sense of [6] (stronger, than bounded completeness).

Definition 4.1. We say that a continuous dcpo A with the basis K is *coherently continuous*, if for any two basic elements $k, k' \in K$, the set of their minimal upper bounds, $\mathrm{MUB}(k,k')$, is finite, and for any $x \in A$, if $k \sqsubseteq x$ and $k' \sqsubseteq x$, then there is $k'' \in \mathrm{MUB}(k,k')$, such that $k'' \sqsubseteq x$.

Theorem 4.1 *If A is a coherently continuous dcpo with countable basis K, then for any $x \in A$, I_x is open.*

Proof. Consider $y \in I_x$. Since the space has a countable basis, we only need to show, that if there is a sequence of basic elements, $k_1 \sqsubseteq k_2 \sqsubseteq \ldots$, such that $y = \sqcup k_i$, then some k_i belongs to I_x.

By contradiction, assume that this is not the case. Then for all i, k_i and x have an upper bound. Using the presence of a countable basis again, approximate x with a sequence of basic elements as well: $l_1 \sqsubseteq l_2 \sqsubseteq \ldots$, $x = \sqcup l_i$. Then for all i, k_i and l_i have an upper bound.

Now we are going to build the sequence of (not necessarily basic) elements, $u_1 \sqsubseteq u_2 \sqsubseteq \ldots$, such that for all i, $k_i \sqsubseteq u_i$ and $l_i \sqsubseteq u_i$. Then $\sqcup u_i$ would be an upper bound of x and y, yielding the desired contradiction.

Consider an element $v \in \mathrm{MUB}(k_i, l_i)$. Define the height of v as maximal j, such that there is $w \in \mathrm{MUB}(k_j, l_j)$, such that $v \sqsubseteq w$. If there is no such maximal natural number, we say that v is of infinite height. Using coherence condition, it is easy to see, that there is an element $u_1 \in \mathrm{MUB}(k_1, l_1)$ of infinite height. Now consider only elements $v \in \mathrm{MUB}(k_2, l_2)$, such that $u_1 \sqsubseteq v$. Using coherence condition, it is easy to see once again, that there is an element $u_2 \in \mathrm{MUB}(k_2, l_2)$ of infinite height, such that $u_1 \sqsubseteq u_2$. Continuing this process, we obtain the desired sequence.

□

Therefore, one can use $N_x = I_x$ in order to obtain all results of the previous section not only for continuous Scott domains, but also for coherently continuous dcpo's with countable bases.

5 When I_x Behaves Badly

5.1 Example

Let start with the example. We define an algebraic countable dcpo E, as a following subset of the powerset of $\mathbf{Z}$, ordered by subset inclusion. $E = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, \{1,2,3,\ldots\}, \{0\}, \{1,0,-1\}, \{1,2,0,-2\}, \{1,2,3,0,-3\}, \ldots\}$. For convenience, we introduce a unique letter denotation for each of the elements of E: $\bot_E = \emptyset$, $e_1 = \{1\}$, $e_2 = \{1,2\}$, $e_3 = \{1,2,3\}, \ldots, e_\infty = \{1,2,3,\ldots\}$, $0_E = \{0\}$, $f_1 = \{1,0,-1\}$, $f_2 = \{1,2,0,-2\}$, $f_3 = \{1,2,3,0,-3\}, \ldots$ We will use this notation throughout the paper. Observe, that all elements, except e_∞, are compact, that elements $e_\infty, f_1, f_2, \ldots$ are total, and that $e_i \sqsubseteq f_j$ iff $i \le j$.

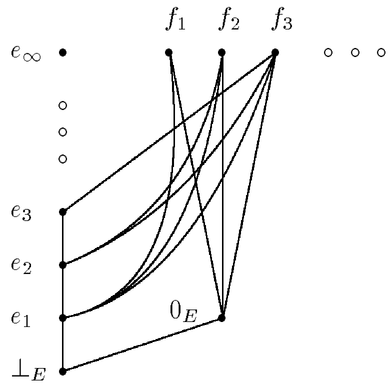


Figure 1: Domain E

Now we will see how *negation duality* works in this example. In our previous terminology, $x = e_\infty$, $y = 0_E$. The role of a directed set B is played by an increasing sequence, $e_1 \sqsubseteq e_2 \sqsubseteq e_3 \sqsubseteq \ldots$. Notice that $e_\infty = \sqcup B$.

Note also that $I_{0_E} = \{e_\infty\}$ and $0_E \in I_{e_\infty}$. You see, that I_{0_E} is not Scott open (although I_{e_∞} is Scott open), and dually, we obtain that $0_E \notin \bigcup_{b \in B} I_b$ (due to observation, that $I_{e_1} = \emptyset$, $I_{e_2} = \{f_1\}$, $I_{e_3} = \{f_1, f_2\}, \ldots$), thus breaking $I_{\sqcup B} = \bigcup_{b \in B} I_b$.

The breaking of this equality prevents the resulting relaxed metrics from being Scott continuous, as 0_E is compact and should naturally carry a finite weight. Since all I_b and $I_{\sqcup B}$ are Scott open, taking $\mathrm{Int}(I_x)$ as N_x instead of I_x would not fix this problem.

5.2 Solution

Let us rewrite the negation duality as $I_x = \{y \mid x \in I_y\}$. What works, somewhat surprisingly, is taking a subset of I_x via taking the interior inside the right-hand side of this expression: $J_x = \{y \mid x \in \mathrm{Int}(I_y)\}$.

Lemma 5.1 *If B is a directed set, $J_{\sqcup B} = \bigcup_{b \in B} J_b$.*

Proof. A potentially non-trivial part is to prove $J_{\sqcup B} \subseteq \bigcup_{b \in B} J_b$. Consider $y \in J_{\sqcup B}$. By definition of J, $\sqcup B \in \mathrm{Int}(I_y)$. Since $\mathrm{Int}(I_y)$ is Scott open, there is $b \in B$, such that $b \in \mathrm{Int}(I_y)$, that is $y \in J_b$.

□

In the next Subsection, we explain this result in terms of Stone duality. In our example domain E, J_{e_∞} does not include 0_E, unlike I_{e_∞}, yielding $J_{e_\infty} = \bigcup_i J_{e_i}$.

In general, J_x is Alexandrov open, but does not have to be Scott open. E.g., in our example domain E, we have that $J_{0_E} = I_{0_E} = \{e_\infty\}$, and thus J_{0_E} is not Scott open. Due to co-continuity we can extend μ to Alexandrov open sets V, by defining $\mu(V) = \mu(\mathrm{Int}(V))$.

However, in order to use a setup of Section 3, it is much more convenient to define $N_x = \mathrm{Int}(J_x)$ and to use the following Lemma.

Lemma 5.2 *If for arbitrary Alexandrov open sets $J, J_m, m \in M$, the equality $J = \bigcup_{m \in M} J_m$ holds, then $\mathrm{Int}(J) = \bigcup_{m \in M} \mathrm{Int}(J_m)$ holds as well.*

Proof. The potentially non-trivial direction is to prove $\mathrm{Int}(J) \subseteq \bigcup_{m \in M} \mathrm{Int}(J_m)$. Consider $y \in \mathrm{Int}(J)$. By the Border Lemma (Lemma 2.1) there is $x \in J$, such that $x \ll y$. Then, because of the condition of the Lemma we are currently proving, there is $m \in M$, such that $x \in J_m$. Then applying the Border Lemma again, we obtain $y \in \mathrm{Int}(J_m)$.

□

Hence, $N_x = \mathrm{Int}(J_x)$ enables us to satisfy all the requirements of the setup of Section 3, except for the requirement that for all $x \in \mathit{Total}(A)$, $C_x \cup N_x = A$, which does not hold in general. E.g. consider our example domain E, and observe that $e_\infty \in \mathit{Total}(A)$, but $0_E \notin C_x \cup N_x$. (Observe, also that changing N_x to J_x does not fix this.)

Thus the Theorem 2.2 holds, but the Theorem 2.3 about the equality of $l(x,y)$ and $u(x,y)$ does not have to hold, and the resulting induced metric on $\mathit{Total}(A)$ cannot in general be obtained. E.g., in our example domain E, we have that $e_\infty \in \mathit{Total}(E)$, but $u(e_\infty, e_\infty) = \mu(I_{e_\infty}) - \mu(J_{e_\infty}) = \mu(\{0_E\})$, which is, in general, not zero, since 0_E is compact.

5.3 Stone Duality

The first Lemma in the previous Subsection holds for the reasons, which are not related to such specific features of J_x as the use of $\mathrm{Int}(I_y)$ (any open set can be used instead) and the fact, that x and y belong to the same set A.

We analyze this situation in the spirit of *Stone duality* [9, 18], which is a contravariant equivalence between categories of spatial frames (of open sets) and sober topological spaces.

For the purpose of this Subsection only, assume that there is a continuous dcpo A (Scott topologies of continuous dcpo's are sober [9]) and a set D, and that we are given a map $U : D \to \mathcal{O}(A)$, where $\mathcal{O}(A)$ is the frame of Scott open sets of the domain A.

Now generalize the construction of J_x by considering the map $J : A \to \mathcal{P}(D)$, where $\mathcal{P}(D)$ is a powerset of D ordered by set-theoretic inclusion and equipped with the Scott topology. Define J by the formula: $J : x \mapsto \{y \in D \mid x \in U(y)\}$. Then observe that the proof of Lemma 5.1 still goes through, implying that J is a Scott continuous function from A to $\mathcal{P}(D)$.

Now observe, if one applies J^{-1} to a subbasic open set $\uparrow\{y\}$, one obtains $J^{-1}(\uparrow\{y\}) = \{x \mid J(x) \in \Uparrow\{y\}\} = \{x \mid y \in J(x)\} = \{x \mid x \in U(y)\} = U(y)$.

Thus the map U can be thought of as defined on the generators $\Uparrow\{y\}$ of the free frame of all Scott open sets on $\mathcal{P}(D)$ and giving raise to the frame homomorphism $u : \mathcal{O}(\mathcal{P}(D)) \to \mathcal{O}(A)$ (of course, $u = J^{-1}$, e.g. for basic open sets, $u(\Uparrow\{y_1, \ldots, y_n\}) = U(y_1) \cap \ldots \cap U(y_n)$, and the similar thing goes for unions of basic sets).

Now, since we are dealing with sober spaces, Stone duality means, that not only $u = J^{-1}$ can be obtained from the continuous function J, but also the continuous function J can be restored from the frame morphism u. And this is the essence of our definition of J, when we think about U as defined on the generators of the frame $\mathcal{O}(\mathcal{P}(D))$.

5.4 J_x Is the Largest Continuous Approximation of I_x

Both I_x and J_x can be considered as functions from A to the powerset of A, $\mathcal{P}(A)$. However, in general, only J_x is Scott continuous. The following theorem shows that, in some sense, J_x is the best we can do.

Theorem 5.1 *If $f : A \to \mathcal{P}(A)$ is a Scott continuous function and $\forall x \in A.\, f(x) \subseteq I_x$, then $\forall x \in A.\, f(x) \subseteq J_x$.*

Proof. Assume that such Scott continuous function f is given, and for some x, there is $y \in f(x)$, such that $y \notin J_x$, i.e. $x \notin \mathrm{Int}(I_y)$. However, $y \in I_x$ means $x \in I_y$. Now consider a directed set $B = K_x$. We have that $x = \sqcup B$ and that all $b \in B$ are way below x. Then, taking into account $x \notin \mathrm{Int}(I_y)$ and applying the Border Lemma, we obtain that $\forall b \in B.\, b \notin I_y$.

However, the assumption of continuity of f means, that $f(\sqcup B) = \bigcup_{b \in B} f(b)$. Hence, since $y \in f(\sqcup B)$, there is some $b \in B$, such that $y \in f(b)$, hence $y \in I_b$, hence $b \in I_y$, contradicting the last formula of the previous paragraph.

□

A similar statement holds for $\mathrm{Int}(I_x)$ and $\mathrm{Int}(J_x)$, understood as functions from A to the dual domain of open sets of A.

6 The Use of the Lawson Condition

We start with the equivalent formulation of the Lawson condition.

Lemma 6.1 *Given a continuous dcpo A, the relative Scott and Lawson topologies on $Total(A)$ coincide if and only if for all $x \in A$, $I_x \cap Total(A) = \mathrm{Int}(I_x) \cap Total(A)$.*

For example, in the domain E from the previous Section Lawson condition does not hold. Indeed, $I_{0_E} = \{e_\infty\}$ and $e_\infty \in Total(E)$. However, $\mathrm{Int}(I_{0_E}) = \emptyset$. The set $\{e_\infty\}$ is open in the relative Lawson topology, but not in the relative Scott topology.

Now we are going to modify the domain E, in order to obtain a different example, which would satisfy the Lawson condition. We add a new element, $\star$, to $\mathbf{Z}$, so that E' will be a subset of the powerset of $\mathbf{Z} \cup \{\star\}$, ordered by the set-theoretic inclusion. Let $E' = E \cup \{e_\star\}$, where $e_\star = \{\star, 1, 2, 3, \ldots\}$.

Now $I_{0_E} = \{e_\infty, e_\star\}$, so this is still not a Scott open set, however, $\mathrm{Int}(I_{0_E}) = \{e_\star\} = I_{0_E} \cap Total(E') = \mathrm{Int}(I_{0_E}) \cap Total(E')$. Notice, that $J_{0_E} = I_{0_E}$ here, so J_{0_E} is still only Alexandrov

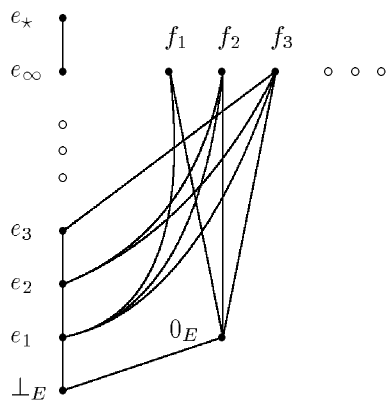


Figure 2: Domain E'

open. J_{e_∞} is the same as in E, but now e_∞ is not a total element. However, $J_{e_\star} = J_{e_\infty} \cup \{0_E\}$, because $e_\star \in \mathrm{Int}(I_{0_E})$, so $J_{e_\star} = I_{e_\star} = \mathrm{Int}(J_{e_\star})$, and $C_{e_\star} \cup J_{e_\star} = C_{e_\star} \cup \mathrm{Int}(J_{e_\star}) = E'$. The set $\{e_\star\}$ is open in both Lawson and Scott relative topologies on $Total(E')$.

What is going on here is described by the following Theorem.

Theorem 6.1 *A continuous dcpo A satisfies the Lawson condition if and only if for all $x \in Total(A)$, $J_x = I_x$. Hence, if the Lawson condition holds, then $C_x \cup J_x = C_x \cup \mathrm{Int}(J_x) = A$.*

Proof. Assume that the Lawson condition holds and $x \in Total(A)$. Assume, that $y \not\sqsubseteq x$, i.e. $y \notin C_x$ and $y \in I_x$, using the totality of x. Thus, by duality, $x \in I_y$. Because $x \in Total(A)$ and because due to the Lawson condition $I_y \cap Total(A) = \mathrm{Int}(I_y) \cap Total(A)$, we obtain $x \in \mathrm{Int}(I_y)$, hence $y \in J_x$.

Conversely, assume $\forall x \in Total(A).\, J_x = I_x$. Let us prove $I_y \cap Total(A) = \mathrm{Int}(I_y) \cap Total(A)$, thus proving the Lawson condition. Take $x \in I_y \cap Total(A)$. By negation duality, $y \in I_x$, then, by totality of x and our assumptions, $y \in J_x$, which, by definition of J_x, means that $x \in \mathrm{Int}(I_y)$.

The rest follows from the observation, that if $x \in Total(A)$, I_x is Scott open.

$\square$

Hence if the Lawson condition holds, the resulting μ*Info*-structure is not deficient, and the Theorem 2.3 holds.

7 Conclusion

We presented a method of obtaining *optimal* Scott continuous lower bounds for relaxed metrics on continuous dcpo's from CC-valuations. For continuous dcpo's with Lawson condition these bounds are exact on total elements, inducing the classical metric structure on total elements. The extension of this method beyond bounded complete continuous dcpo's was possible via the use of negation duality and Stone duality considerations.

It might be useful to extend the Stone duality analysis to $\mathrm{Int}(J_x)$ and to be able to speak about the intuition behind the Lawson condition in the spirit of [16].

We would also like to ask the following question. If Lawson condition does not hold, can we obtain some negative results about the existence of μ*Info*-structures with totality property, or, more generally, about the existence of relaxed metrics with the property $\forall x, y \in \mathit{Total}(A).\, l(x,y) = u(x,y)$? Obviously, this question allows a number of variations, e.g. we know now, that when this question is restricted to the case of $\mathit{Info}(x) = C_x$, such negative results can indeed be obtained.

Acknowledgements

We would like to thank Steve Matthews, Simon O'Neill, Joshua Scott, Mike Smyth, Steve Vickers, and Julian Webster for helpful discussions. We are in debt to the influence of Klaus Keimel, who encouraged us to leave the familiar safety of bounded completeness. We are especially thankful to Bob Flagg for focusing our attention on the Lawson condition.

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