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Partial Metrics, Fuzzy Equalities, and Metric-Entropy Pairs

June 2009

Slides for this talk:

http://www.cs.brandeis.edu/∼bukatin/partial-metrics-talk-jun-2009.pdf

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Mathematics of partially defined elements.

Generalized distances: instead of $p(x, x) = 0$ axiom, value $p(x, x)$ expresses how far x is from being completely defined.

Generalized equalities: instead of $x = x$ being always true, value $= (x, x)$ expresses how well defined x is.

$$p(y,y) + p(x,z) \leq p(x,y) + p(y,z)$$

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

$$q(x,y) = p(x,y) - p(x,x)$$

$$d(x,y) = q(x,y) + q(y,x) = 2p(x,y) - p(x,x) - p(y,y)$$

1. Semantics of programming languages. Generalized metrization of non-Hausdorff topologies.

2. Sheaves and fuzzy sets.

3. Entropy of partitions.

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Measuring common information.

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1. Semantics of programming languages. Dana Scott

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Example: interval numbers

Consider segments $[a, b]$ and $[c, d]$.

Define the distance between them as

$$max(b, d) - min(a, c).$$

Example: partially defined functions

The degree of equality of two functions f and g is the interior of $\{x \in X \mid f(x) = g(x)\}$.

Domains for denotational semantics

(Dana Scott)

Partial order, Scott topology, Scott continuous functions.

Sierpinski space: the domain for **void**

Two-element space: $\{\mathbf{undefined}, \mathbf{result}\}$

Using Sierpinski space as an example:

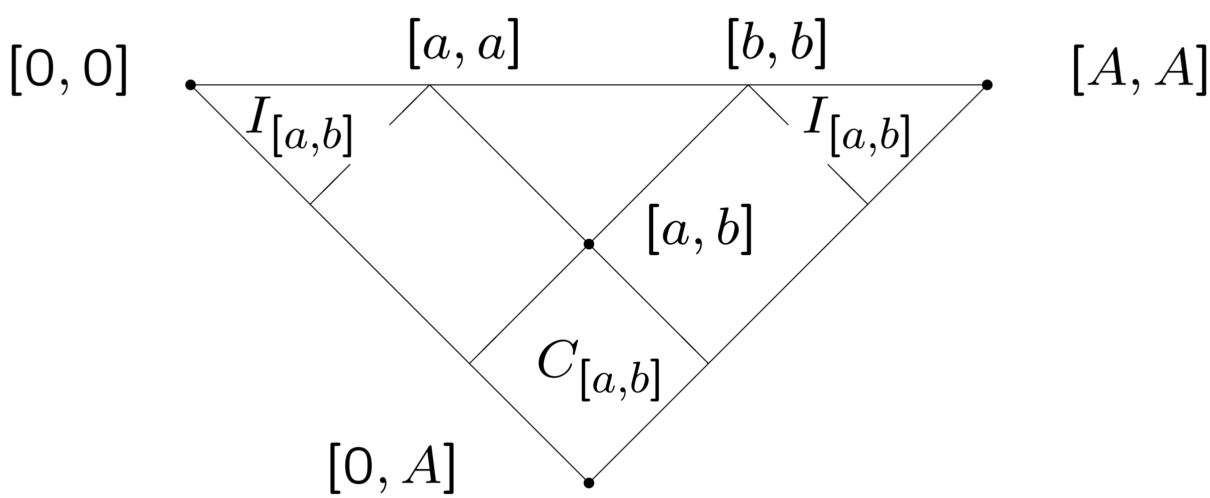
“Tobin Bridge distance” vs. **partial metric**

Both allow us to define Scott topology.

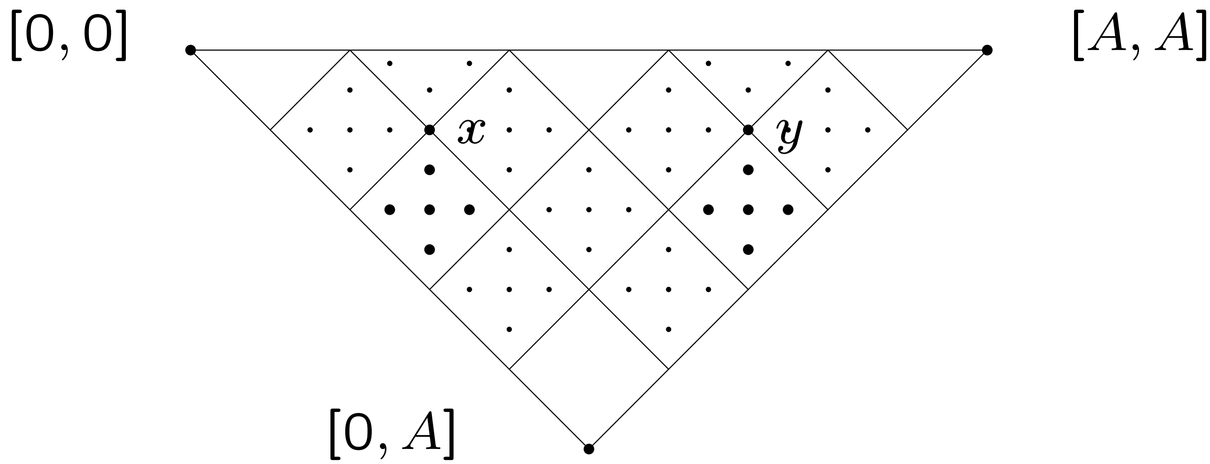
Only partial metric is Scott continuous.

Partial metrics via measures of common information

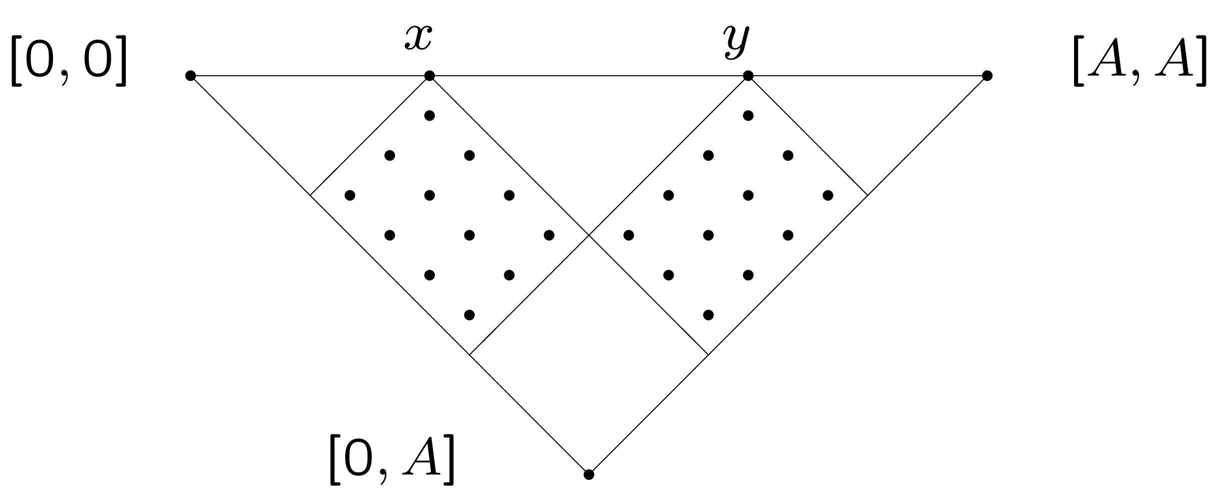
Define C_x as a closed set of all segments, containing segment x, and I_x as an open set of all segments, not intersecting with segment x.



We can informally think, that C_x represents all positive information known about x, and I_x represents all negative information known about x, i.e. such information which cannot become true when the partially defined number x gets defined better.



Observe, that for totally defined numbers, $x = [a, a]$ and $y = [b, b]$, upper and lower bounds coincide, and a classical metric results.



Steve Matthews: **partial metrics**

$p(x, y) = p(y, x)$ (symmetry)

$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$

$p(x, x) \leq p(x, y)$ (small self-distance)

$p(y, y) + p(x, z) \leq p(x, y) + p(y, z)$ (strong triangularity aka **Vickers' triangularity**)

late 1980s – early 1990s (originally came from the need to prove the absence of deadlock in lazy data-flow)

`http://partialmetric.org`

Weighted quasi-metrics

Non-negative weight $w(x)$

$w(x) + q(x, y) = w(y) + q(y, x)$

Given partial metric $p(x, y)$, we define:

$q(x, y) = p(x, y) - p(x, x)$ and $w(x) = p(x, x)$.

Given a weighted quasi-metric (q, w) we define:

$p(x, y) = w(x) + q(x, y) = w(y) + q(y, x)$.

Weighted metrics

Non-negative weight $w(x)$

$w(x) - w(y) \leq d(x, y)$

Given partial metric $p(x, y)$, we define:

$d(x, y) = q(x, y) + q(y, x) = 2p(x, y) - p(x, x) - p(y, y)$

and $w(x) = p(x, x)$.

Given a weighted metric (d, w) we define:

$p(x, y) = \frac{d(x,y)+w(x)+w(y)}{2}$.

Metrics with the base point

Application: approximating classical metrics

Ω-sets

Ω-valued fuzzy equalities

D.Scott,M.Fourman,D.Higgs (1970s)

Ω – complete Heyting algebra

complete lattice, $\sqsubseteq$

for all a, b, there is greatest x, denoted as $a \to b$, such that $a \wedge x \sqsubseteq b$.

A topology is a typical complete Heyting algebra: $\sqsubseteq = \subseteq$, $\wedge = \& = \cap$, $U \to V = \text{Int}(V \cup \bar{U})$.

Ω-valued fuzzy equality: $E : A \times A \rightarrow \Omega$

Axioms:

$E(a,b) = E(b,a)$

$E(a,b) \wedge E(b,c) \sqsubseteq E(a,c)$

Partial ultrametrics, $p(x,z) \leq max(p(x,y), p(y,z))$, can be viewed as fuzzy equalities.

$[0, +\infty]$ can be thought of as the Scott topology on positive reals.

$\sqsubseteq = \subseteq = \geq$ (order is reversed)

Fourman and Scott also introduced a mechanism of *singletons*, which was used to define the notion of *complete Ω-set* and to establish that complete Ω-sets and sheaves over complete Heyting algebra Ω are essentially the same thing.

[see Fourman M. P., D. S. Scott, *Sheaves and Logic*, in “Applications of Sheaf Theory to Algebra, Analysis, and Topology,” Lecture Notes in Mathematics, **753**, Springer, 1979, pp. 302–401.]

Quantale-valued partial metrics

R.Kopperman, S.Matthews, H.Pajoohesh (2004)

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

- $p(x, x) \leq p(x, y)$
- $p(x, y) = p(y, x)$
- $p(x, z) \leq p(x, y) + (p(y, z) \dot{-} p(y, y))$

Quantale-valued sets

Quantale-valued fuzzy equalities

Ulrich Hoehle (early 1990s)

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

An M-valued set is a set X equipped with a map $E : X \times X \to M$ (**fuzzy equality**) subject to the axioms

- $E(x, y) \sqsubseteq E(x, x)$
- $E(x, y) = E(y, x)$
- $E(x, y) * (E(y, y) \Rightarrow E(y, z)) \sqsubseteq E(x, z)$

We noticed the equivalence between partial metrics and fuzzy equalities in 2006:

http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html

Metric-entropy pairs

Dan Simovici

Metric-Entropy Pairs on Lattices, Journal of Universal Computer Science (Springer-Verlag), vol. **13**, no.11, 2007, pp. 1767-1778

http://www.cs.umb.edu/∼dsim/papersps/de.pdf

Definition 1, formula (1): The pair (d, η) is a $\wedge$-pair if $d(x, y) = 2\eta(x \wedge y) - \eta(x) - \eta(y)$.

Theorem 4, formula (3): Given a $\wedge$-pair (d, η), axiom $d(x, y) \leq d(x, z) + d(z, y)$ holds if and only if $\eta(z) + \eta(x \wedge y) \leq \eta(x \wedge z) + \eta(y \wedge z)$ for all x, y, z.

Section 3. Conditional function of a $\wedge$-pair (d, η) is defined as $\kappa(x, y) = \eta(x \wedge y) - \eta(y)$.

Consider $p(x, y) = \eta(x \wedge y)$.

Then $p(x, x) = w(x) = \eta(x)$.

We also have $\kappa(x, y) = q(y, x)$.