Self-modifying Dynamical Systems: a Primer

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A simple example of self-modifying dynamical system described in Appendix D.2.2 of our `https://arxiv.org/abs/1610.00831` preprint.

Let us describe the construction of **lightweight pure dataflow matrix machine**. We consider rectangular matrices $M \times N$. We consider discrete time, $t = 0, 1, \ldots$, and we consider $M + N$ streams of those rectangular matrices, $X^1, \ldots, X^M, Y^1, \ldots, Y^N$. At any moment t, each of these streams takes a rectangular matrix $M \times N$ as its value. (For example, X_t^1 or Y_t^N are such rectangular matrices. Elements of matrices are real numbers.)

Let's describe the rules of the dynamical system which would allow to compute $X_{t+1}^1, \ldots, X_{t+1}^M, Y_{t+1}^1, \ldots, Y_{t+1}^N$ from $X_t^1, \ldots, X_t^M, Y_t^1, \ldots, Y_t^N$. We need to make a choice, whether to start with $X_0^1, \ldots, X_0^M$ as initial data, or whether to start with $Y_0^1, \ldots, Y_0^N$. Our equations will slightly depend on this choice. In our series of preprints we tend to start with matrices $Y_0^1, \ldots, Y_0^N$, and so we keep this choice here, even though this might be slightly unusual to the reader. But it is easy to modify the equations to start with matrices $X_0^1, \ldots, X_0^M$.

Matrix Y_t^1 will play a special role, so at any given moment t, we also denote this matrix as A. Define $X_{t+1}^i = \sum_{j=1,\ldots,N} A_{i,j} Y_t^j$ for all $i = 1, \ldots, M$. Define $Y_{t+1}^j = f^j(X_{t+1}^1, \ldots, X_{t+1}^M)$ for all $j = 1, \ldots, N$.

So, $Y_{t+1}^1 = f^1(X_{t+1}^1, \ldots, X_{t+1}^M)$ defines Y_{t+1}^1 which will be used as A at the next time step $t+1$. This is how the dynamical system modifies itself in lightweight pure dataflow matrix machines.

Example similar to the one from Appendix D.2.2

Define $f^1(X_t^1, \ldots, X_t^M) = X_t^1 + X_t^2$. Start with $Y_0^1 = A$, such that $A_{1,1} = 1$, $A_{1,j} = 0$ for all other j, and maintain the condition that first rows of all other matrices $Y^j, j \neq 1$ are zero. These first rows of all $Y^j, j = 1, \ldots, N$ will be invariant as t increases. This condition means that $X_{t+1}^1 = Y_t^1$ for all $t \geq 0$.

Let's make an example with 3 constant update matrices: Y_t^2, Y_t^3, Y_t^4. Namely, say that $f^2(X_t^1, \ldots, X_t^M) = U^2, f^3(X_t^1, \ldots, X_t^M) = U^3, f^4(X_t^1, \ldots, X_t^M) = U^4$. Then say that $U_{2,2}^2 = U_{2,3}^3 = U_{2,4}^4 = -1$, and $U_{2,3}^2 = U_{2,4}^3 = U_{2,2}^4 = 1$, and that all other elements of U^2, U^3, U^4 are zero[1]. And imposing an additional starting condition on $Y_0^1 = A$, let's say that $A_{2,2} = 1$ and that $A_{2,j} = 0$ for $j \neq 2$.

Now, if we run this dynamic system, the initial condition on second row of A would imply that at the $t = 0$, $X_{t+1}^2 = U^2$. Also $Y_{t+1}^1 = X_{t+1}^1 + X_{t+1}^2$, hence now taking $A = Y_1^1$ (instead of $A = Y_0^1$), we obtain $A_{2,2} = 1 + U_{2,2}^2 = 0$, and in fact $A_{2,j} = 0$ for all $j \neq 3$, but $A_{2,3} = U_{2,3}^2 = 1$.

Continuing in this fashion, one obtains $X_1^2 = U^2, X_2^2 = U^3, X_3^2 = U^4, X_4^2 = U^2, X_5^2 = U^3, X_6^2 = U^4, X_7^2 = U^2, X_8^2 = U^3, X_9^2 = U^4, \ldots$, while the invariant that the second row of matrix Y_t^1 has exactly one element valued at 1 and all other zeros is maintained, and the position of that 1 in the second row of matrix Y_t^1 is 2 at $t = 0$, 3 at $t = 1$, 4 at $t = 2$, 2 at $t = 3$, 3 at $t = 4$, 4 at $t = 5$, 2 at $t = 6$, 3 at $t = 7$, 4 at $t = 8$, ...

This element 1 moving along the second row of the network matrix is a simple example of a circular wave pattern in the matrix $A = Y_t^1$ controlling the dynamical system in question.

It is easy to use other rows of matrices U^2, U^3, U^4 as "payload" to be placed into the network matrix Y_t^1 for exactly one step at a time, and one can do other interesting things with this class of dynamical systems.

[1] Essentially we are saying that those matrices "point to themselves with weight -1", and that "U^2 poiints to U^3, U^3 points to U^4, and U^4 points to U^2 with weight 1".