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On Duality between Metric and Logical Viewpoints

Joint work with Ralph Kopperman and Steve Matthews

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Slides for this talk:

http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html

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To memory of Denis Higgs

Duality between metric and logical viewpoints

The metric viewpoint: how far two objects are from each other.
The logical viewpoint: to what degree two objects overlap.

Fuzzy mathematics is traditionally done from the logical viewpoint, so the first step in introducing fuzzy metrics is often the transformation f(x,y) = exp(-d(x,y)) (SumTopo2009).

Then we have the following correspondences:

$d(x,y) = 0$ if and only if $f(x,y) = 1$.

$d(x,y) = +\infty$ if and only if $f(x,y) = 0$.

$d(x_1,y_1) < d(x_2,y_2)$ if and only if $f(x_2,y_2) < f(x_1,y_1)$.

The axiom $d(x,x) = 0$ becomes $f(x,x) = 1$.

The axiom $d(x,z) \leq d(x,y) + d(y,z)$ becomes $f(x,y) * f(y,z) \leq f(x,z)$.

Non-expansive maps become maps which respect overlap by not letting it decrease.

Etc..

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$d(x_1, y_1) < d(x_2, y_2)$ if and only if ~~$f(x_2, y_2) < f(x_1, y_1)$~~ $f(x_2, y_2) \sqsubset f(x_1, y_1)$.

The axiom $d(x, x) = 0$ becomes $f(x, x) = 1$.

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Duality between metric and logical viewpoints

What if we omit the exponent: $f(x, y) = -d(x, y)$?

Then we have the following correspondences:

$d(x, y) = 0$ if and only if $f(x, y) = 0$.

$d(x, y) = +\infty$ if and only if $f(x, y) = -\infty$.

$d(x_1, y_1) < d(x_2, y_2)$ if and only if $f(x_2, y_2) \sqsubset f(x_1, y_1)$.

The axiom $d(x, x) = 0$ becomes $f(x, x) = 0$.

The axiom $d(x, z) \leq d(x, y) + d(y, z)$ becomes
$f(x, y) + f(y, z) \sqsubseteq f(x, z)$.

Non-expansive maps become maps which respect overlap by not letting it decrease.

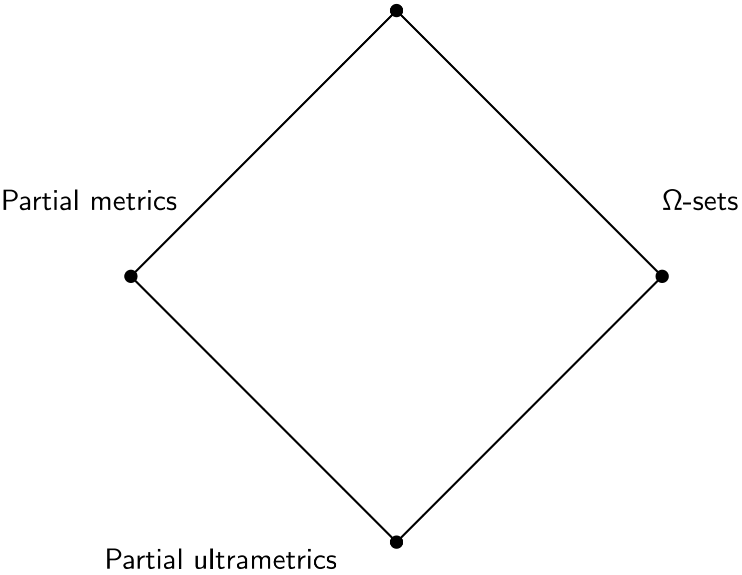
Reviewing duality between logical and metric viewpoint for partially defined elements

Mathematics of partially defined elements

Generalized distances: instead of $p(x,x) = 0$ axiom,
value $p(x,x)$ expresses how far x is from being completely defined.

Generalized equalities: instead of $x = x$ being always true,
value $=(x,x)$ expresses how well defined x is.

Quantale-valued partial metric = Quantale-valued sets



Example: interval numbers

Consider segments $[a, b]$ and $[c, d]$ on the real line.
Define the distance between them as

$$max(b, d) - min(a, c).$$

partial metrics (Steve Matthews)

$p : X \times X \to \mathbf{R}^+$

$p(x, y) = p(y, x)$ (symmetry)
$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$
$p(x, x) \leq p(x, y)$ (small self-distance)
$p(y, y) + p(x, z) \leq p(x, y) + p(y, z)$ (strong triangularity (Steve Vickers))

Some applications of partial metrics

Good for generalized metrization of a wide class of (usually non-Hausdorff) topologies and bi-topologies.

Main case for computer science so far: the Scott topology.

In the bitopological setting: dually, lower topology, and Lawson topology as join of Scott and lower.

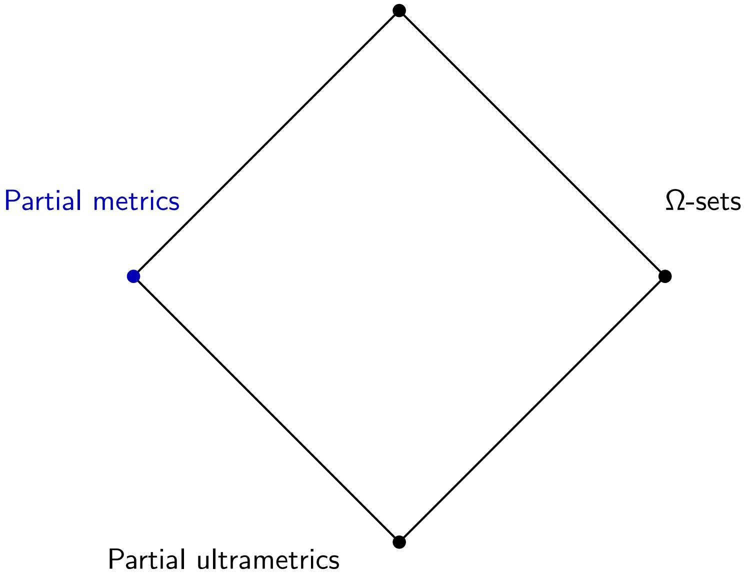
Computer scientists love them, because one can obtain Scott continuous generalized metrizations of Scott domains with partial metrics, when one adopts a logical viewpoint of $\mathbf{R}^+$.

This cannot be done with quasi-metrics, because of $q(x, x) = 0$.

A logical viewpoint of R^+

$\sqsubseteq = \geq$ and we equip this set with a Scott topology.

Quantale-valued partial metric = Quantale-valued sets



Example: partially defined functions

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

The degree of equality of two functions (f, U) and (g, V) is the interior of $\{x \in U \cap V \mid f(x) = g(x)\}$.

Fuzzy equalities valued in complete Heyting algebras

Ω-sets

Ω-valued fuzzy equalities

D.Scott,M.Fourman,D.Higgs (1970s)

complete Heyting algebras

Ω – complete Heyting algebra

complete lattice, $\sqsubseteq$

for all a, b, there is greatest x, denoted as $a \to b$, such that $a \wedge x \sqsubseteq b$.

A topology is a typical complete Heyting algebra: $\sqsubseteq = \subseteq$, $\wedge = \& = \cap$, $U \to V = \mathrm{Int}(V \cup \bar{U})$.

Sets valued in complete Heyting algebras

Ω-valued fuzzy equality: $E : A \times A \to \Omega$

Axioms:

$E(a, b) = E(b, a)$

$E(a, b) \wedge E(b, c) \sqsubseteq E(a, c)$

Example (again): partially defined functions

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

The degree of equality of two functions (f, U) and (g, V) is the interior of $\{x \in U \cap V \mid f(x) = g(x)\}$.

Another example: pre-sheaves of sets over Ω

Let x be a section over a, and y - a section over b.

$E(x, y) = \bigvee\{p \sqsubseteq a \wedge b \mid x \upharpoonright_p = y \upharpoonright_p\}$.

Distributivity of complete Heyting algebras is used to establish $E(a, b) \wedge E(b, c) \sqsubseteq E(a, c)$.

Some applications: singleton completion and sheafification

Singletons are functions $s : A \to \Omega$, such that

$s(x) \sqsubseteq E(x,x)$

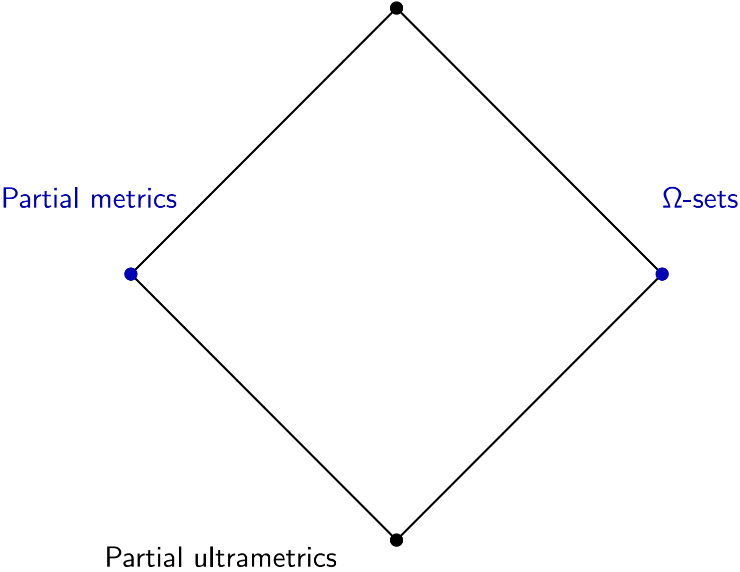
$s(x) \wedge E(x,y) \sqsubseteq s(y)$

$s(x) \wedge s(y) \sqsubseteq E(x,y)$

All functions $y \mapsto E(x,y)$ are singletons. If all singletons are uniquely represented in this way, the Ω-set is called complete. Otherwise, an Ω-set can be mapped into the set of its singletons, resulting in singleton completion.

Complete Ω-sets with morphisms $f : (A,E) \to (B,F)$ satisfying $E(x,x) = F(f(x), f(x))$ and $E(x,y) \sqsubseteq_B F(f(x), f(y))$ are equivalent to the category of sheaves of sets over Ω.

Quantale-valued partial metric = Quantale-valued sets



partial ultrametrics

$p(x,y) = p(y,x)$ (symmetry)
$p(x,x) = p(x,y) = p(y,y) \Rightarrow x = y$
$p(x,z) \leq max(p(x,y), p(y,z))$

Example: consider set of finite and infite sequences over some alphabet, and denote the length of the common prefix of sequences s_1 and s_2 as $L(s_1, s_2)$. Then define $p(s_1, s_2) = 2^{-L(s_1, s_2)}$.

Partial ultrametrics are partial metrics

$p(x,x) \leq max(p(x,y), p(y,x)) = p(x,y)$, obtaining small self-distances.

$p(x,z) \leq max(p(x,y), p(y,z))$ means $p(x,z) \leq p(x,y)$ or $p(x,z) \leq p(y,z)$. Consider $p(x,z) \leq p(x,y)$.

We know that $p(y,y) \leq p(y,z)$, so $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$, obtaining strong triangularity.

A logical viewpoint of R^+ revisited

$[+\infty, 0]$ can be thought of as the Scott topology on positive reals, $(0, +\infty)$.

x corresponds to the $(x, +\infty)$ Scott open set, so 0 corresponds to the whole space $(0, +\infty)$, and $+\infty$ corresponds to the empty open set.

$\sqsubseteq = \subseteq = \geq$ (order is reversed)

Partial ultrametrics are Ω-sets

Consider $p(x, z) \leq max(p(x, y), p(y, z))$.

Consider $[+\infty, 0]$ as the Scott topology on positive reals.

Then "$p(x, y)$" $\wedge$ "$p(y, z)$" $\sqsubseteq$ "$p(x, z)$".

Or, writing this in details,
$(p(x, y), +\infty) \cap (p(y, z), +\infty) \subseteq (p(x, z), +\infty)$

Partial ultrametrics are Ω-sets

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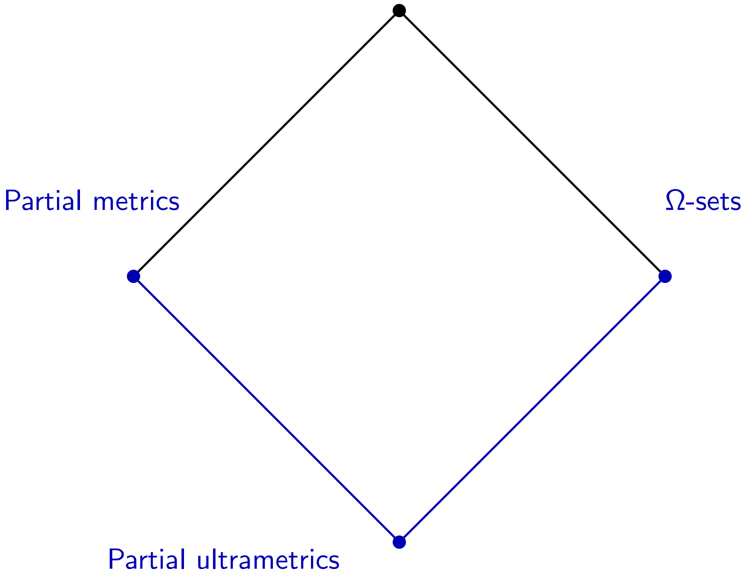
Or, writing this in details,
$(p(x,y), +\infty) \cap (p(y,z), +\infty) \subseteq (p(x,z), +\infty)$

In general, whenever I talk about Ω-sets from a metric viewpoint, and, thus, have $p(x,z) \leq p(x,y) \vee p(y,z)$, I tend to call p a partial ultrametric valued in the system of numbers in question.

Quantale-valued partial metric = Quantale-valued sets



Quantale generalizations

Quantale-valued partial metrics

R.Kopperman, S.Matthews, H.Pajoohesh (2004)

used for generalized metrization of large topologies

Quantale-valued sets

Quantale-valued fuzzy equalities

Ulrich Hoehle (early 1990s)

used for bringing substructal logic to fuzzy sets, e.g. non-idempotent conjunctions such as Lukasiewicz conjunction:

$x \& y = max(0, x + y - 1)$.

Partial pseudometrics into quantales

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

Quantale-valued sets

An M-valued set is a set X equipped with a map $E : X \times X \to M$ (**fuzzy equality**) subject to the axioms

- $E(x,y) \sqsubseteq E(x,x)$
- $E(x,y) = E(y,x)$
- $E(x,y) * (E(y,y) \Rightarrow E(y,z)) \sqsubseteq E(x,z)$

Quantales

Metric viewpoint

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

Logical viewpoint

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Partial pseudometrics into quantales

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

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Quantales

Metric viewpoint

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a + c \geq b\}$. Certain additional conditions are imposed.

Logical viewpoint

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Do we have a duality between partial metrics and fuzzy equalities?

Specialization orders are the same in both cases (better defined elements are higher).

There are many choices of the notion of morphism, giving various interesting categories, but in all cases we know morphisms go in the same direction yielding (covariant) equivalences (even isomorphisms) of the categories in question.

So it seemed to me in 2009 that it was correct to talk about equivalence between partial metrics and fuzzy equalities up to the choice of notation and not about duality between them. After all, every time we have a partial order, we also have a dual partial order on the same set, and what we have here is just the choice which of these two orders to use to express the same axioms.

The duality in the choice of notation (and in spirit) could have been handled on a meta-level (and, could have been, perhaps, left informal).

Do we have a duality between partial metrics and fuzzy equalities?

Mustafa Demirci disagreed with us and told us that we should talk about duality here, and that we should not go to the meta-level, but should use the formal duality between truth values and distances values to express this duality (e.g. if the truth values formed a Heyting algebra, then the distance values were to form a dual Heyting algebra, etc).

Do we have a duality between partial metrics and fuzzy equalities?

I think the right compromise here is as follows. On one hand we should emphasize that partial metrics and fuzzy equalities are really the same up to a dual viewpoint, that we have isomorphisms between their various classes, that no contravariance was discovered here so far.

On the other hand, the duality of the viewpoints should not be left informal and should not be formalized on the meta-level, but, as Demirci suggested, should be formalized via duality between truth values and distance values, and should be emphasized as strongly as possible, because it really reflects the duality between intuitions of mathematicians working on the metric side of things and on the logical side of things.

Emphasizing duality between logical and metric viewpoints as strongly as possible

If truth values form a Heyting algebra (e.g. an algebra of open sets), we should not say that the correspondent distance values form a dual Heyting algebra. We should say instead that the distance values form a Browerian algebra (e.g. an algebra of closed sets, not an algebra of open sets with reversed order).

If distance values are non-negative reals, we should not say that the corresponding truth values are non-negative reals turned upside down (like everyone does, and like we did in the preceeding slides). We should say instead that the truth values are non-positive reals.

Then one would avoid the annoying need to "turn one's head upside down", which one usually has to do all the time in this context.

The ultrametric case and sheaves

Consider a complete Heyting algebra Ω. Consider a corresponding complete Browerian algebra α.

Every separated pre-sheaf of sets over complete Heyting algebra Ω can be thought of as a separated co-pre-sheaf of α-ultrametrics and non-expansive maps (a bit of technical work is required to prove this).

For example, consider a sheaf of partially defined functions.

Sheaf of partially defined functions (we've seen this)

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

The degree of equality of two functions (f, U) and (g, V) is the interior of $\{x \in U \cap V \mid f(x) = g(x)\}$.

For $V \subseteq U$, restriction $(f, U) \upharpoonright_V$ is (f', V), where for any $x \in V, f'(x) = f(x)$.

The partial ultrametric here is the compliment to the fuzzy equality, that is a closure of the set of elements where either $f(x) \neq g(x)$, or either $f(x)$, or $g(x)$, or both are undefined, with self-distance from (f, U) to itself being just a complement of U.

Co-sheaf of closed-set-valued ultrametric spaces and non-expansive maps

The closed-set-valued (non-partial) ultrametric on a section associated with $\bar{U}$ is

$$p((f, U), (g, U)) \dot{-} \bar{U} = \text{Closure}(\{x \in U \mid f(x) \neq g(x)\}).$$

For $V \subseteq U$, the associated co-restriction from $\bar{U}$ to $\bar{V}$ produces $p((f', V), (g', V)) \dot{-} \bar{V} = \text{Closure}(\{x \in V \mid f(x) \neq g(x)\}) \subseteq \text{Closure}(\{x \in U \mid f(x) \neq g(x)\})$, hence non-expansiveness of the restriction map.

Partial metrics into non-negative reals

Their logical counter-part should "morally" be fuzzy equalities valued in non-positive reals.

Partial ultrametrics correspond to idempotent logic (usually, to the ordinary intuitionistic logic). Partial metrics should typically correspond to the linear logic, and we think about linear logic as the resource-sensitive logic.

So, from the linear logic point of view, it is natural to think about the weight (self-distance) of an element as the work which still needs to be done to make it fully defined. This is the work to be done, something owed, hence negative (like in our balance statements).

Relaxed metric considerations

Relaxed metric typically maps (x, y) into an interval number $[l(x, y), u(x, y)]$, where u is usually a partial metric, and l is usually a symmetric function, such that $l(x, y) \leq u(x, y)$.

u yields an upper bound for the inequality of "true, underlying x and y"; essentially, "x and y differ no more that $u(x, y)$", while l yields a lower bound for that, essentially, "x and y differ at least by $l(x, y)$". There is an intimate relationship between l and negative information, and also between l and tolerances.

From the earlier logical considerations of relaxed metrics (starting with [Bukatin and Joshua Scott, 1997]) we know that u dualizes, but l does not. This means that on the logical side, U becames negative (non-positive, actually), but L remains non-negative.

Relaxed metric considerations

So, while U represents a work still owed (a work to estimate distance better, actually), and hence negative, L represents a work done, and hence positive (on the logical side).

Interestingly enough, the condition $l(x, y) \leq u(x, y)$ on the metric side becomes $L(x, y) + U(x, y) \leq 0$ on the logical side.

If the distance between 2 elements, x and y, is precesely defined (often the case for maximal elements x and y), then $l(x, y) = u(x, y)$, or equivalently $L(x, y) + U(x, y) = 0$, expressing the fact that no further computations are owed.

In general the amount which expressed debt here is not $U(x, y)$, but $L(x, y) + U(x, y) = l(x, y) - u(x, y)$. (Note that $l(x, x)$ is always 0, so the self-distance is always fully owed.)

Space-time considerations

The distance which still needs to be covered is owed, is debt, is negative.

The distance which we already covered is an asset, a positive.

With time it is the opposite. The time spent is resource spent, it's gone, it's negative. The time ahead is a resource we can use, it's positive.

A CATEGORY APPROACH TO BOOLEAN-VALUED SET THEORY

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AUGUST, 1973

(= order-ideal etc. in a poset) of $\mathcal{B}$ and that, for x in S, ε gives the inverse to the bijection $[0,a]\to[0,x]$. F is said to be étale if the set of sections of F has join 1. Let $\mathcal{E}$ be the category of CHA's and étale mappings. Then $\mathcal{E}^{op}/\mathcal{A}$ is equivalent to $S(\mathcal{A})$ for each CHA $\mathcal{A}$.

To give an indication of why this is so, let $\mathcal{A}\xrightarrow{F}\mathcal{B}$ be étale and let S be the set of sections of F. Then (S,δ) is a sheaf where $\delta(x,y) = \varepsilon(x\wedge y)$ ((S,δ) is clearly an $\mathcal{A}$-valued set and one shows that $x \leq y$ in (S,δ) if and only if $x \leq y$ in $\mathcal{B}$, and that $\{x_i\}_{i\in I}$ is a compatible family in (S,δ) if and only if $\bigvee_i x_i$ is in S). Conversely, given a sheaf (S,δ) on $\mathcal{A}$, let $\mathcal{B} = P(S,\delta)$ and define $\mathcal{A}\xrightarrow{F}\mathcal{B}$ by $a \mapsto a\wedge\varepsilon$; then F is étale. It is straightforward to turn these constructions into functors and one way to show that they are inverse to each other to within natural isomorphism is to use essentially the same argument as was used in proving Theorem 2 of [11] (wherein, we now see, only injective $\mathcal{A}$-valued sets with $\mathcal{A}$ boolean were considered, such $\mathcal{A}$-valued sets corresponding to étale, there called analytic, embeddings $\mathcal{A}\to\mathcal{B}$).

By modifying its definition somewhat, the functor from $\mathcal{E}^{op}/\mathcal{A}$ to $Sh(\mathcal{A})$ may be extended to a functor from $\mathcal{H}^{op}/\mathcal{A}$ to $Sh(\mathcal{A})$, where $\mathcal{H}$ is the category of CHA's and <u>all</u> $\wedge,\bigvee$-preserving mappings, and it results that $\mathcal{E}^{op}/\mathcal{A}$ is a coreflective subcategory of $\mathcal{H}^{op}/\mathcal{A}$ (all of this being just as it is in the topological case when our CHA's are lattices of open sets).

D. $S(\mathcal{A})$ for $\mathcal{A}$ an arbitrary site

Insofar as the use of $\mathcal{A}$-valued sets sometimes provides an economical description of sheaves on $\mathcal{A}$, the question arises whether the same sort of construction is possible when $\mathcal{A}$ is an arbitrary site (a <u>site</u>

here being a small category with a Grothendieck topology on it): it is possible, but at some cost in simplicity.

Let $\mathcal{A}$ be a site. If a and b are objects of $\mathcal{A}$, let $\mathcal{A}/(a, b)$ denote the category of diagrams

$= (r, s)$.

Then an $\mathcal{A}$-valued set is a triple (X, ε, δ) where X is a set, ε is a function from X to the set of objects of $\mathcal{A}$, δ is a function on $X \times X$ such that, for all x, y in X, $\delta(x, y)$ is a crible in $\mathcal{A}/(\varepsilon(x), \varepsilon(y))$ and the following conditions are satisfied for all x, y, z in X and appropriate r, s, t in $\mathcal{A}$:-

(i) $(r,r) \in \delta(x,x)$,

(ii) $(r,s) \in \delta(x,y)$ implies $(s,r) \in \delta(y,x)$,

(iii) $(r,s) \in \delta(x,y)$ and $(s,t) \in \delta(y,z)$ implies $(r,t) \in \delta(x,z)$,

(iv) if $\{p; (rp, sp) \in \delta(x,y)\}$ covers then $(r,s) \in \delta(x,y)$.

A morphism $(X, \varepsilon, \delta) \xrightarrow{f} (Y, \varepsilon, \delta)$ is a function f on $X \times Y$ such that, for all x in X and y in Y, $f(x,y)$ is a crible in $\mathcal{A}/(\varepsilon(x), \varepsilon(y))$ and, for all x's and r's etc., we have

(v) $(r,r') \in \delta(x,x')$ and $(r,s) \in f(x,y)$ implies $(r',s) \in f(x',y)$,
$(r,s) \in f(x,y)$ and $(s,s') \in \delta(y, y')$ implies $(r,s') \in f(x,y')$,

(vi) if $\{p; (rp,sp) \in f(x,y)\}$ covers then $(r,s) \in f(x,y)$,

(vii) $(r,s) \in f(x,y)$ and $(r,s') \in f(x,y')$ implies $(s,s') \in \delta(y,y')$,

(viii) $\{r;$ there exist y and s such that $(r,s) \in f(x,y)\}$ covers.

Let $(X, \varepsilon, \delta) \xrightarrow{f} (Y, \varepsilon, \delta)$ and $(Y, \varepsilon, \delta) \xrightarrow{g} (Z, \varepsilon, \delta)$ be morphisms. Then $(X, \varepsilon, \delta) \xrightarrow{gf} (Z, \varepsilon, \delta)$ is defined by the condition that $(r,t) \in (gf)(x,z)$ if and only if
$\{u;$ there exist y and s such that $(ru,s) \in f(x,y)$ and $(s,tu) \in g(y,z)\}$
covers.

The study of the resulting category $S(\mathfrak{A})$ goes much as before, though complicated by the fact that we have no joins but have to use covering cribles instead. In particular, it is still the case that $S(\mathfrak{A})$ is equivalent to the category $\tilde{\mathfrak{A}}$ of sheaves on $\mathfrak{A}$ (and there is still a fairly obvious functor $\hat{\mathfrak{A}} \longrightarrow S(\mathfrak{A})$ which, combined with this equivalence, yields the sheafifying functor $\hat{\mathfrak{A}} \longrightarrow \tilde{\mathfrak{A}}$).

E. 'Boolean' powers and ultrapowers

Powering by $\mathfrak{A}$: $S \xrightarrow{\wedge} S(\mathfrak{A}) \xrightarrow{\Gamma} S$, and ultrapowering by $(\mathfrak{A}, \mathcal{U})$; $S \xrightarrow{\wedge} S(\mathfrak{A}) \xrightarrow{c_{\mathcal{U}}} S(\mathfrak{A})_{\mathcal{U}} \xrightarrow{\Gamma} S$, are both well-known and we confine ourselves to a few remarks. (Some references are [5] [6] [10] and [16]; in [6] various more general constructions are discussed.)

For a logician, one of the first questions about these constructions and the functors they involve is: What logical notions do they preserve? (They all preserve monos so the question makes good sense - for the external logical operations, that is). Any geometrical morphism $\mathcal{T}' \rightarrow \mathcal{T}$ of topos with $\mathcal{T}'$ boolean, and in particular $S \xrightarrow{\wedge} S(\mathfrak{A})$ here, preserves all first-order logic (see G.E. Reyes [19]; also R. Mansfield [16], Corollary 1.2, for the case of $S \xrightarrow{\wedge} S(\mathfrak{A})$ with

This is more like a weighted metric in spirit. (Weighted ultrametric, that is.)

What we really want to do here is to try to see whether this generalizes to Lawvere-Tierney topology. This might be easier intuitively than to study Grothendieck topology directly, and it might be easier to dualize, which we would need to do, if we want to go to a metric viewpoint (what Higgs did here is from the logical viewpoint).

Lawvere-Tierney topology is a categorical generalization of a frame nucleus (a closure operation preserving intersections). If we succeed in doing what the previous paragraph suggests, and if we want to generalize this beyond idempotent/ultrametric case and towards quantales and linear logic, the first thing to try is to see whether one can start with a quantic nucleus and get an appropriate generalization of Lawvere-Tierney topology and work with that.

An accidental discovery I made yesterday: pouring coffee into a cup with a tea bag produces a nice coffee drink (it tastes softer than pure coffee, but stimulates stronger).

Slides for this talk:

`http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html`

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