Analytic Properties of Approximation-Based Models of Computations: a Tutorial

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1. Introduction

Denotational semantics of a programming language is provided by a function

$$\textbf{Programs} \to \textbf{Space of meanings}.$$

Such function maps a program to a meaning, which expresses and abstracts computational behavior of this program.

The most popular approach to denotational semantics of programming languages is associated with Dana Scott and Christopher Strachey. Under this approach, the space of meaning is endowed with an *approximation* relation, which is a relation of partial order. This relation induces *Scott topology*, and all computable functions are represented as continuous functions with respect to this topology.

1.1. Introduction continued

For the introduction to the subject see [M.Bukatin. *Continuous Functions as Models for Programs: Mathematics and Applications*, PhD Thesis Proposal, 1996] on my home page and references therein.

These continuity motives tempt researchers to develop analogies of structures encountered in ordinary continuous mathematics, and especially in functional analysis, and to explore potential applications of methods of classical mathematics to software engineering.

During the last several years I managed to organize a group of researchers, who achieved considerable progress towards these goals. Here I illustrate main ideas behind our results on the simple example of *interval numbers*.

2. Interval Numbers

Interval numbers are real-line segments. Segment $[a, b]$ is interpreted as a partially defined number x, about which it is known that

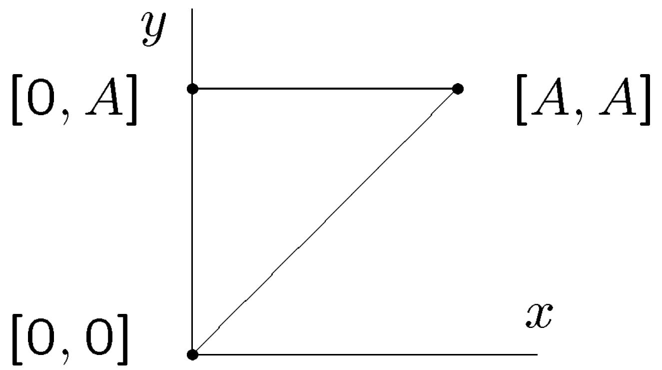
$$a \leq x \leq b.$$

If $x = [a, b]$, we call a a *lower bound* of x, and b an *upper bound* of x. For simplicity, we consider the situation where all numbers are between 0 and some positive number, A.

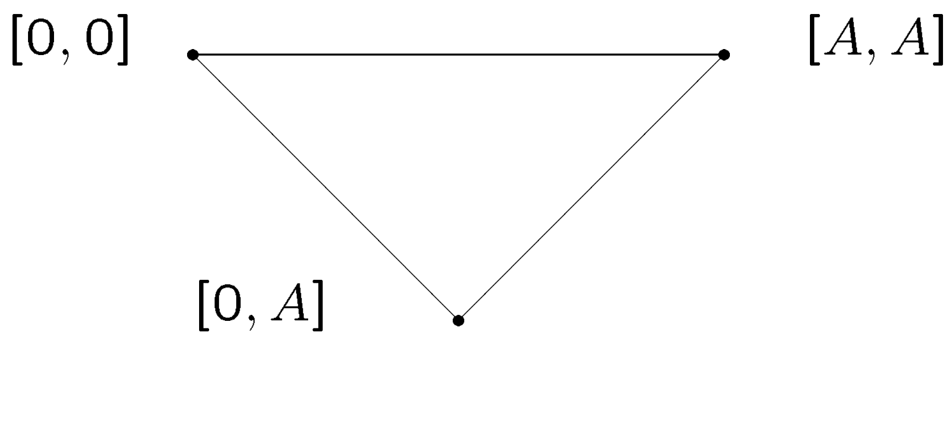
If interval $[c, d]$ is within interval $[a, b]$, that is $a \leq c \leq d \leq b$, than $[c, d]$ gives more information about a number, than $[a, b]$. In such case, we say that $[a, b]$ *approximates* $[c, d]$, and write $[a, b] \sqsubseteq [c, d]$.

2.1. Interval Numbers continued

Interval numbers $[x, y]$ can be thought of as a triangle on the plane.



We flip this triangle so that the best defined interval numbers, like $[a, a]$, are on the top, and the least defined one, $[0, A]$, is on the bottom.



3. Relaxed Metrics

Denote the set of interval numbers in question as R^I.

Since interval numbers are only partially defined, the distances between them should also be defined only partially, i.e. we want to talk only about *estimates* of the true distances.

In particular, even if x and y are two equal intervals, the upper bound of the distance between the associated numbers is generally not zero. E.g., $x = y = [1, 3]$ just means $1 \leq x \leq 3$ and $1 \leq y \leq 3$, hence upper bound of the distance between x and y is 2.

3.1. Relaxed Metrics continued

When two intervals do not intersect, it is also possible to give a non-zero lower bound of the distance. In general, there is a number of ways to define interval distances between interval numbers. We consider

$$\rho([a,b],[c,d]) = [l,u],$$

where

$$u = u([a,b],[c,d]) = max(b,d) - min(a,c)$$

and

$$l = max(0, c-b, a-d).$$

Function ρ is an example of a *relaxed metric* — a symmetric distance function to interval numbers, where upper bounds obey classical triangle inequality, but where $\rho(x,x) = 0$ axiom in general does not hold.

4. Open Balls and Relaxed Metric Topology

An *open ball* with a center $x \in R^I$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in R^I \mid u(x,y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those partially defined numbers y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

A subset U of R^I is called *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > u(x,x)$ such that $B_{x,\epsilon} \subseteq U$.

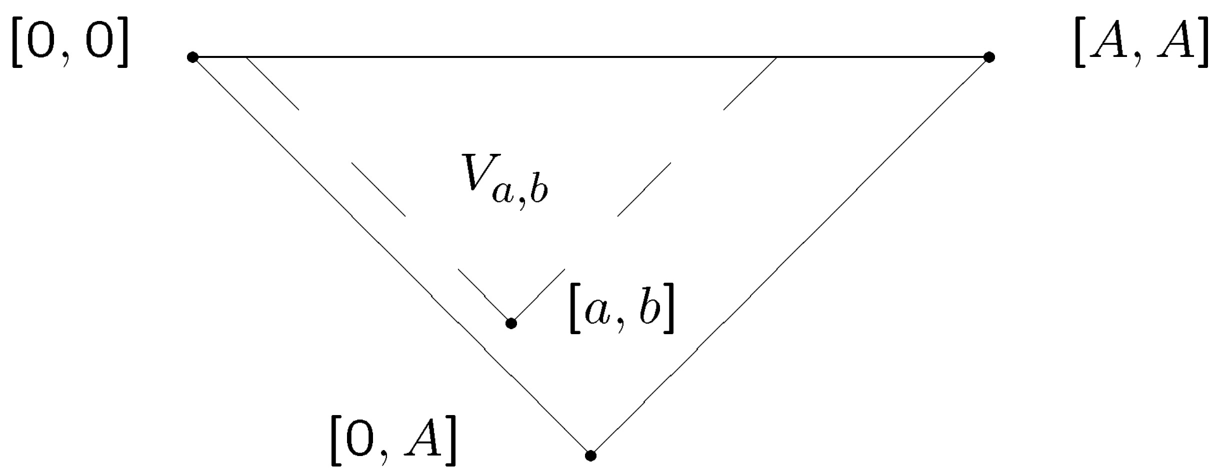
5. Scott Topology and Scott Continuity

In a moment we will characterize relaxed metric open sets for the distance ρ in question and note that they form *Scott topology*.

The distances without axiom $\rho(x, x) = 0$ were first introduced by Steve Matthews under the name of *partial metrics*. We have shown that the rejection of this axiom is essential for any approach where distance functions are computable, explained computational meaning of these distances by generalizing them to *relaxed metrics*, and built the relaxed metrics which yield Scott topology for a large class of approximation domains. See [M.Bukatin, J.Scott. Towards computing distances between programs via Scott domains. *Lecture Notes in Computer Science*, **1234**, 33–43, Springer, 1997] (also on my home page) and references therein.

5.1. Scott Topology continued

First observe, that sets $V_{a,b} = \{[x, y] \mid a < x \leq y < b\}$ are relaxed metric open.



Any relaxed metric open set U can be obtained via sets $V_{a,b}$ in the following fashion: $U = \bigcup_{[a,b]\in U} V_{a,b}$. In fact, it is enougn to use only rational a and b. From this one can see, that relaxed metric open sets form a topology, which has a countable basic consisting of sets $V_{a,b}$ with rational a and b, $0 \leq a < b \leq A$.

5.2. Scott Topology continued

This topology coincides with *Scott topology*. Instead of giving the standard definition of Scott topology, we only characterize here functions $f : R^I \to R^I$, which are continuous with respect to this topology.

These are precisely the functions possessing the property of monotonicity, $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$, and the property of preservation of limits of informationally increasing sequences, which can be thought of as the property of computational schemes involved being *well-defined* with respect to approximations. Consider $a_1 \leq a_2 \leq \ldots \leq b_2 \leq b_1$. Then $[\sup a_i, \inf b_i]$ is the exact informational upper bound of sequence $[a_1, b_1] \sqsubseteq [a_2, b_2] \sqsubseteq \ldots$ We require that $f([\sup a_i, \inf b_i])$ be the exact upper bound of sequence $f([a_1, b_1]) \sqsubseteq f([a_2, b_2]) \sqsubseteq \ldots$

6. Co-continuous Measures

Since areas of $V_{a,b}$ are well-defined, and since they can be used to form a countable basis for our topology, we can define measure $\mu(U)$ of an open set U as its area. This measure possesses two remarkable properties, *strong non-degeneracy* and *co-continuity*.

Strong non-degeneracy means that for open sets $U \subset V \Rightarrow \mu(U) < \mu(V)$. That is, every addition to an open set increases its area.

Co-continuity means, that for any decreasing sequence of open sets, $U_1 \supseteq U_2 \supseteq \ldots$, $\mu(\mathrm{Int}(\bigcap U_i)) = \inf \mu(U_i)$. This means, that the border, $\bigcap U_i \setminus \mathrm{Int}(\bigcap U_i)$, has measure zero.

Co-continuous measures were introduced and built for a large class of domains by our group. Later Klaus Keimel generalized our result and noted that co-continuous measures are impossible for ordinary real line.

6.1. Co-continuous Measures continued

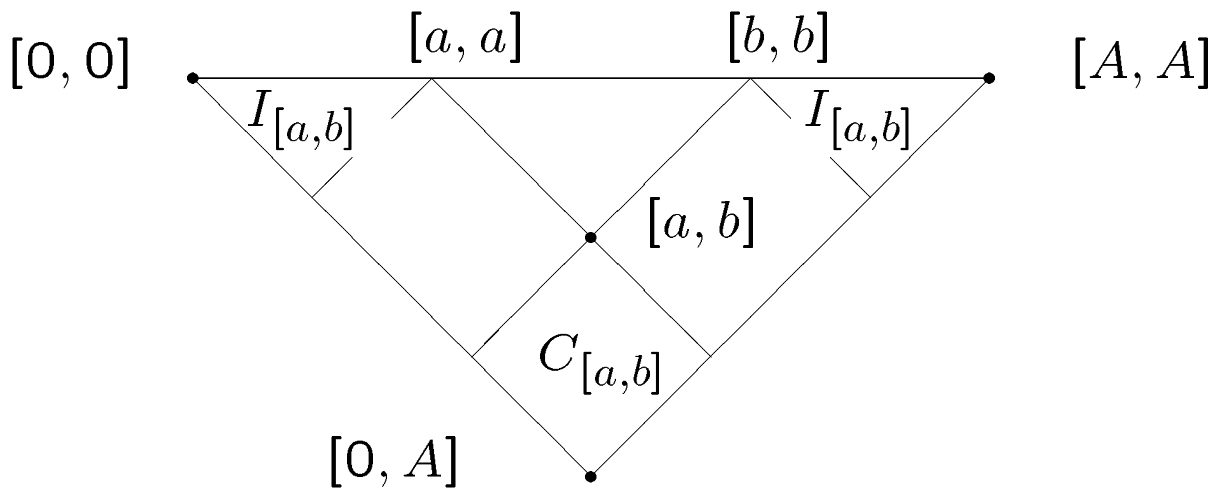
Measures on ordinary real line, however, would exhibit both strong non-degeneracy and co-continuity, if instead of considering arbitrary open sets in both conditions, one would only consider open intervals.

This indicates that Scott topology is coarse enough for its system of open sets to exhibit behaviors similar to the behaviors of typical *bases of open sets* of Hausdorff topologies. This is probably the reason for the absense of many pathologies known as "counterexamples in analysis" in our field.

For more information see [M. Bukatin, S. Shorina. *Partial metrics and co-continuous valuations*. To appear at FoSSaCS'1998] on my home page and references therein.

7. Relaxed Metrics via Co-continuous Measures

Define C_x as a closed set of all segments, containing segment x, and I_x as an open set of all segments, not intersecting with segment x.



We can informally think, that C_x represents all positive information known about x, and I_x represents all negative information known about x, i.e. such information which cannot become true when the partially defined number x gets defined better.

7.1. Metrics via Measures continued

We define relaxed distance between segments x and y, $\rho(x,y) = [l(x,y), u(x,y)]$, based on the idea, that the more of common positive and negative information is known about x and y, the smaller can the upper bound of the distance between them be made. On the other hand, the information, which is positive about one partially defined number and negative about another, contributes to the lower bound of this distance.

Hence, given a measure μ on R^I, we write

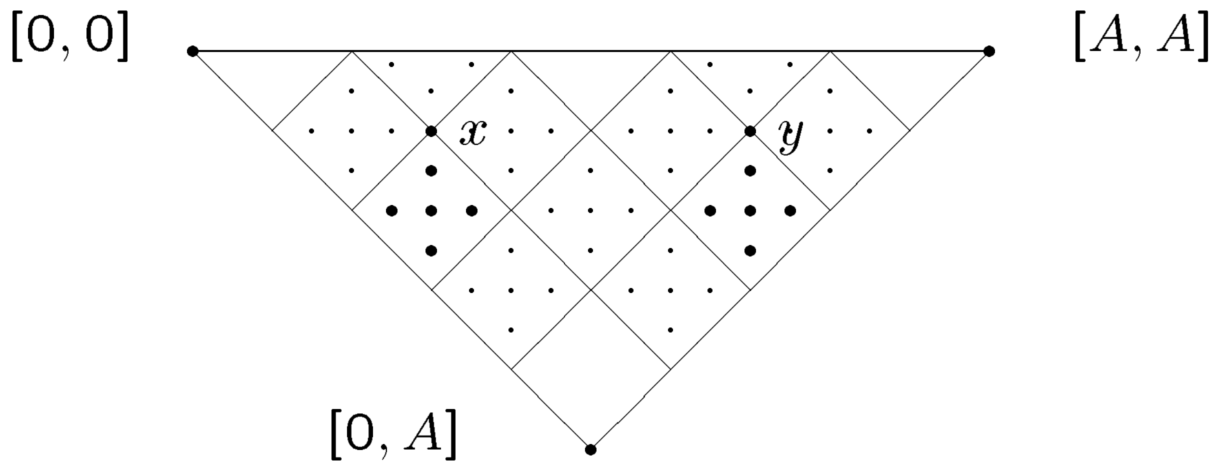
$$u(x,y) = \mu(R^I) - \mu(C_x \cap C_y) - \mu(I_x \cap I_y),$$

and

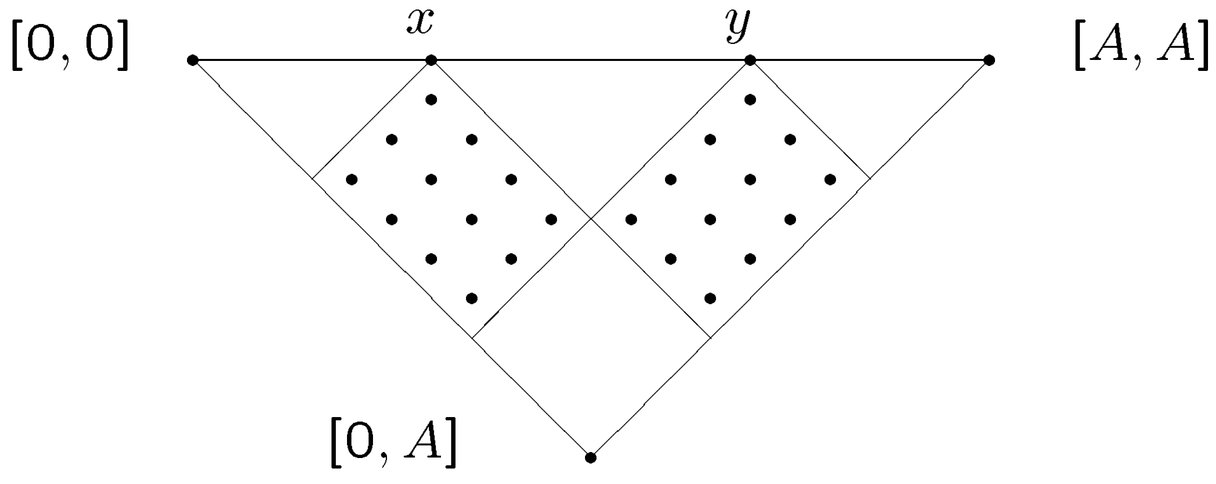
$$l(x,y) = \mu(C_x \cap I_y) + \mu(C_y \cap I_x).$$

On the next page the area with large dots accounts for $l(x,y)$, and the area with all dots (large and small) accounts for $u(x,y)$.

7.2. Metrics via Measures continued



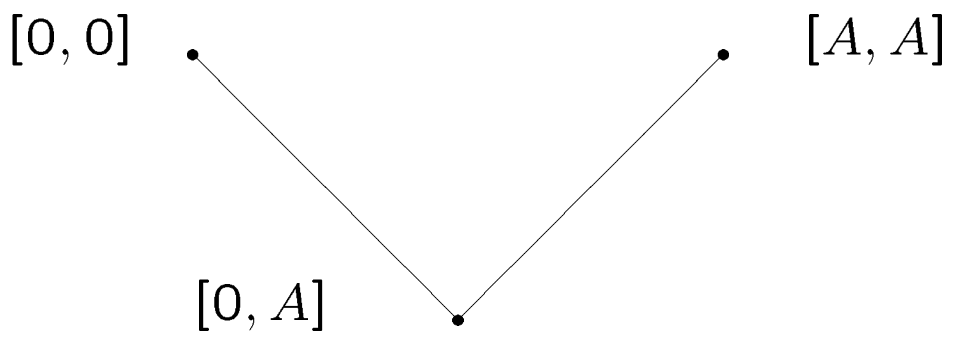
Observe, that for totally defined numbers, $x = [a,a]$ and $y = [b,b]$, upper and lower bounds coincide, and a classical metric results.



7.3. Metrics via Measures continued

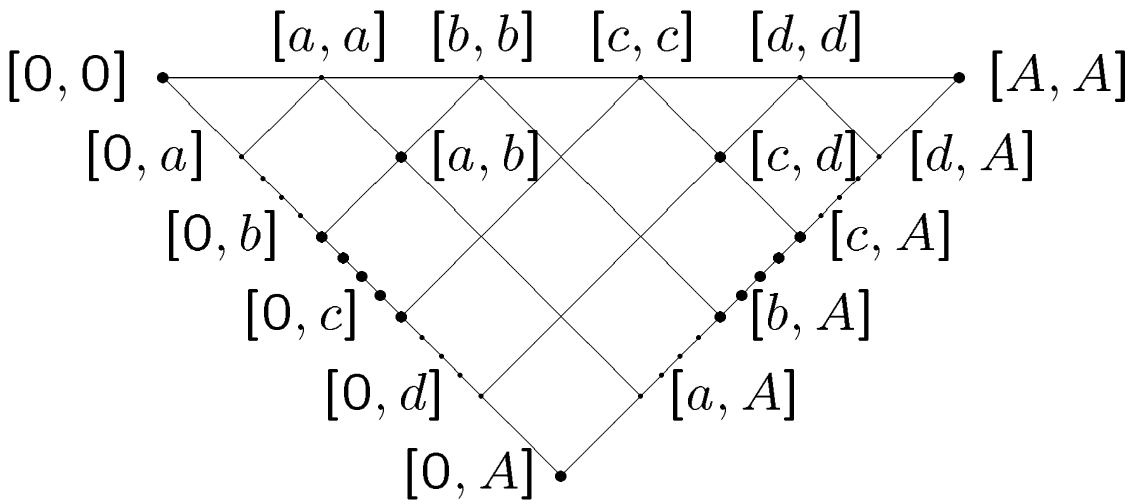
The measure must be co-continuous in order to insure Scott continuity of resulting relaxed metric. It is not so clear what conditions are needed to ensure that the induced relaxed metric topology and Scott topology are equal. We know that if μ is strongly non-degenerate, then the equality of topologies results.

However, the distance ρ from Section 3 can be obtained from a measure, which is not strongly non-degenerate, but instead is all carried by the two sides of the triangle:



7.4. Metrics via Measures continued

Specifically, we define the measure of a Borel subset of R^I as one-half of the length of its intersection with these two sides. Then the analysis of the following picture will help to make sure that ρ indeed results.



8. Conclusion: Inverse Problems

While a lot of open questions in the analysis on approximation domains remain, there can be little doubt that the rapid development of the field will continue in the coming years.

However, it seems that in order to obtain truly innovative applications to software engineering, we need to learn to solve a class of problems, which can be thought of as the **inverse problem** to the well-studied and solved problem of representing programs via continuous functions. This is the problem of how to find reasonable approximations to given Scott continuous functions, which would be sufficiently efficient from the practical point of view.

8.1. Conclusion continued

More specifically, we would like to learn to efficiently approximate certain domain elements on the basis of their description in a specified formal language. In the setting of denotational semantics, the formal language in question is not the programming language itself, but the language used to write formulas of denotational semantics.

We see two classes of possible approaches here. One class of approaches suggests the use of the underlying recursively enumerable structure via methods of inductive inference and learning, algorithmic probability, and Kolmogorov complexity. Another class of approaches suggests the consideration of this problem as a classical problem from the conventional numerical mathematics, similar to the problem of approximating a solution of a variational problem — it might be fruitful not to consider the underlying recursive enumerations explicitly under this class of approaches.